Marketing via Friends: Strategic Diffusion of Information in Social Networks with Homophily

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Abstract

The paper studies the impact of homophily on the optimal strategy of a monopolist, whose marketing campaign of a new product relies on word of mouth communication. Homophily is a tendency of people to interact more with those who are similar to them. In the model there are two types of consumers embedded into a social network, which differ in friendship preferences and desirable product design. Consumers are engaged in word of mouth communication and can learn about the product directly from advertisement or from their neighbors. The monopolist chooses the product design and price to influence the pattern of communications. We find that for low levels of homophily compromise design of the product is preferred to specialized products, even if there is no cost of producing more than one product. Moreover, an increase of homophily raises price elasticity of demand and benefits both the monopolist and consumers. Finally, we show that a product attractive to both types may be optimal even though the monopolist only profits from sales to one type.

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1 Introduction

In the last decade word of mouth (WOM hereafter) also known as viral marketing has received a considerable amount of attention from mass-media and scientific community as efficient marketing tool (see for instance Campbell, 2009, Goyal and Galeotti, 2007, Leskovec et al. 2007, and Iribarren and Moro, 2007). The WOM marketing takes an

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advantage of a natural human's inclination to spread information. A recent study by Reichheld (2003) shows the pervasive importance of consumers' communication about the product for companies perspectives. In particular, they show that the willingness of consumers to recommend a company to their friends by far is the best predictor of the company's growth.

Along with striking successes of some WOM campaigns, the majority of them even fails to meet more moderate aim, such as to induce multiple sales per advertisement. A report by Riley and Wigder (2007) from Jupiter Research reveals that only 15% of viral campaigns are considered to be successful. Moreover, 55% of companies planned to reduce the use of this tactics next year. There can be many reasons for this to happen which are beyond the scope of our paper. For example, the product itself may not be appealing enough to consumers or companies may choose a wrong strategy for advertisement, etc.

In the paper we argue that the performance of WOM campaign crucially depends on the role that the product plays in the social interactions of consumers. If product characteristics serve as some kind of identity upon which consumers build their relationships then consumers that value the same characteristics will end up connected more often. This tendency of people to interact more with those who are similar to them is known as "homophily" and has been documented at least since Aristotle's time.[1,2]

The description of the model is the following. There is a monopolist that introduces a new product to an initially uninformed population of heterogenous consumers of two types. Consumers are embedded into a social network, which is represented by a random graph. Across types, consumers differ in friendship preferences and desirable design of the product. Within the types, consumers differ in a willingness to pay for the product. We model consumers friendship preferences by a linking bias towards the same type of consumers, which represents homophily of the society. Consumers communicate with their friends and learn about the existence of the product from neighbors who already have *acquired* it.

The monopolist knows statistical properties of a network such as degree distribution and homophily level and strategically chooses price and design of the product. To induce sales we assume that the monopolist advertises the product directly to an infinitesimal part of the population and the rest of the population is expected to find out about the product through a WOM communication.

Our analysis begins by examining a necessary conditions on a network structure such that WOM can spread over a significant proportion of the population[3]. This was a case

[1]In Aristotles Rhetoric and Nichomachean Ethics, he noted that people "love those who are like themselves" (Aristotle 1934, p. 1371).

[2]The term homophily appeared in the sociological literature for the first time in Lazarsfeld and Mertons (1954) who also quoted the proverbial expression - "birds of a feather flock together," which has summarized homophily ever since.

[3]In the network literature this phenomenon is known as a global cascade

of such remarkable examples of WOM campaigns as diffusion of Hotmail accounts[4] and the advertising campaign of tiny budget movie "The Blair Witch Project"[5]. We find that in the case of sparse networks a sufficiently high level of homophily is a necessity for a success of WOM campaign. High levels of homophily imply that preferences of connected consumers are correlated, which allows the monopolist to develop the product attractive for longer chains of connected consumers.

Next, we turn to the optimal design of the product. In the diffusion literature the commonly employed assumption is that a message to be spread in a network is given, and the main focus is upon the effect of network structure on its propagation (for a survey see Geroski, 2000). In contrast, we assume an active role of the monopolist. In the model the monopolist designs a message by choosing the price and characteristic of the product. In our base-line model we find that only two types of design are optimal - compromise and a specialized one. Thus for a sufficiently high level of homophily specialized products designed to target needs of one type of consumers are optimal. However, when the homophily level is sufficiently low, the compromise design is preferred even when there is no cost of producing more than one type of product. The latter happens, since the majority of links connect consumers of different types and to insure spreading the product should be attractive to both types.

The sociological literature on homophily adopts a view that diversity of individual's contacts is a socially desirable property per se (e.g. Moody, 2001). Although this assertion could be supported by evidence, no rigorous analysis has been made. Perhaps surprisingly, in our model social welfare is increasing in the level of homophily. The result comes from informational and monetary benefits for consumers generated by an increase in the level of homophily. Informational benefits consist in a higher awareness of consumers about the product. Monetary benefits come from a lower price charged by the monopolist, which converts a higher awareness of the product into a higher volume of sales.

There is a popular idea in business and academic literature that focusing advertisement efforts on a group of consumers is the efficient strategy. We show that it is indeed true - an advertisement strategy of targeting consumers of one group is optimal. However, the same does not always hold for the product design, even when advertisement is targeted to one group. In the case when the society exhibits low levels of homophily the optimality of product specialization depends on the density of the social network. If the density is low then the expected demand triggered per advertisement is small and it is optimal to specialize on a group of consumers targeted by the advertisement. Nevertheless, if the network is sufficiently dense it is optimal for the monopolist to choose compromise design. The latter strategy sacrifices some initial adopters from the targeted group, but insures higher subsequent spread.

[4]Hotmail spent a mere 50,000 dollars on traditional marketing and still grew from zero to 12 million users in 18 months.

[5]A movie released in 1999 with principal photography budget ranging from $20,000 to $25,000.

A term "freakonomics" firmly entered vocabularies of many economists. The popular book of the same name[6] with over 3 million copies sold worldwide provoked numerous discussions in academic circles, while the primary audience was general public[7]. Influenced by this phenomenon, we consider the monopolist, which is interested only in one type of consumer (for instance the academic community). We show that designing a product attractive to both types of consumers may be the optimal strategy, even though monopolist benefits only from one type. In other words writing "freakonomics" type of a book may be a good strategy to become famous in academic circles.

There are two streams of networks literature closely related to our paper. The first one studies strategic diffusion of information assuming that nodes are matched randomly (see, for instance, Campbell, 2009, Goyal and Galeotti, 2007, Galeotti and Mattozzi, 2008). The second stream considers the impact of homophily on various processes unfolding on networks (for instance Golub and Jackson, 2009, Buhai at el, 2008, and Valat, 2009).

The two recent papers from diffusion literature most relevant to our research are Campbell (2009) and Goyal and Galeotti (2007). Campbell (2009) studies the optimal pricing and advertisement strategies of a monopolist when consumers are engaged in WOM communication. Goyal and Galeotti (2009) study general model of the strategic diffusion, where they explicitly distinguish between the level and content of the interaction. In their paper, the level of interaction is characterized by a degree distribution, while content of interaction is a way in which actions of others affect individual incentives.

In the framework of WOM literature our paper contributes in two dimensions. First and most importantly, the paper introduces homophily into the network upon which WOM spreads and studies its impact on the optimal strategy of the monopolist. The notion of homophily enriches network structure by specifying a probability of friendship relationships among groups of consumers. Second, the paper extends the monopolist's problem by including product design that affects further WOM communication. To the author's best knowledge product design has not been the subject of academic research in WOM framework.

The recent paper from the second stream of literature Golub and Jackson (2009) studies how different mechanisms of communication operating through a network are affected by homophily of the society. The principal difference of our paper is that in our setup the monopolist (the sender of a message) takes an active role and influences WOM spreading by choosing "message" to spread on the network.

The rest of the paper is organized as follows. Section 2 presents a stylized model of strategic diffusion. In section 3 we derive the expected size of cascade of sales per advertisement. Section 4 presents the main results on the optimal price and design of the

[6]The full title of the book: "Freakonomics: A Rogue Economist Explores the Hidden Side of Everything".

[7]In other domains one can recall examples such as "Linked: The New Science of Networks" on networks by Barabási, "The Selfish Gene" on evolution by Richard Dawkins etc.

product and considers welfare implications of homophily. Section 5 considers the effect of network density on the optimal strategy of the monopolist. Section 6 examines the optimal product design and advertisement strategy when the monopolist can target advertisement by types of consumers. Section 7 considers an example when the monopolist is interested only in one group of consumers. Section 8 studies robustness of the obtained results to a variation in the shape of preference frontier and considers the case of a global cascade of sales. Finally, Section 9 outlines avenues for future research and concludes.

2 Model

The model, consists of three main blocks: network structure, consumer preferences and monopolist problem.

2.1 Consumer Preferences

There is a continuum of consumers of two types A and B, which are embedded into an undirected social network. Consumers of type A constitute measure γ of the population and the rest are consumers of type B. We focus on the case with consumers of two types because it provides basic intuitions and insights, while keeping the analysis transparent[8].

In their purchasing behavior consumers differ in two respects. First, across types consumers differ in preferences towards a product design. Consumers of type A prefer one characteristic of the product, while consumers of type B are interested in the opposite feature. Second, within the types consumers differ in a reservation price $\bar{P}_j$ they are willing to pay for the product and the minimal level of desirable characteristic $\bar{w}_j$, which induces them to buy the product.

More formally, in the model two variables affect the decision of consumers to buy the product: the price $P \in [0, 1]$ and the characteristic of product $w \in [0, 1]$. For concreteness, a consumer j of type A buys the product if characteristic is higher than the threshold level $w \geq \bar{w}_j$ and the price is lower than the reservation price $P \leq \bar{P}_j$. In contrast, a consumer l of type B buys the product if $w \leq \bar{w}_l$ and $P \leq \bar{P}_l$.

We assume that within a type the reservation price and characteristic threshold are distributed according to $f^i(\bar{w}, \bar{P})$ probability density function. Hence, a randomly chosen consumer j of type A, which is aware of the product with a characteristic w and price P buys it with probability:

$$q^A = Pr(w \geq \bar{w}_j \cap P \leq \bar{P}_j) = \int_0^w \int_P^1 f^A(\bar{w}, \bar{P}) d\bar{w} d\bar{P}$$

[8]For example, the case of consumers of three types with the third type that is not interested in the product is the same as the case of two types with a corrected degree distribution.

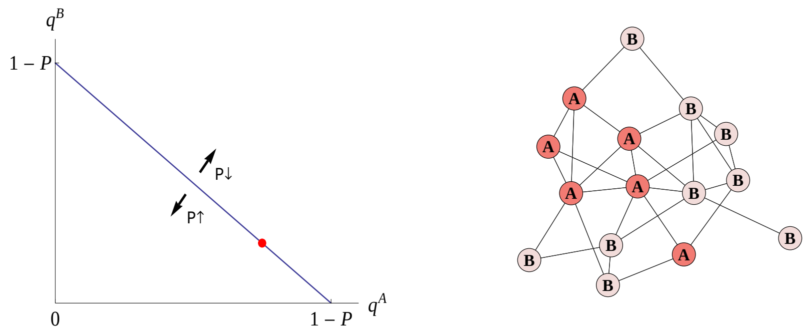


Figure 1: On the left hand side preferences frontier with characteristic of the product being marked by circle. On the right hand side implied social network, with probability to buy the product shown by the intensity of color.

And similarly a randomly selected consumer l of type B, which knows about the product buys it with the probability:

$$q^B = Pr(w \leq \bar{w}_l \cap P \leq \bar{P}_l) = \int_w^1 \int_P^1 f^B(\bar{w}, \bar{P}) d\bar{w} d\bar{P}$$

To simplify the analysis for the major part of it we assume that the threshold characteristic and threshold price distributed independently and identically according to the uniform distribution $U[0, 1]$ for both types. This implies that $f^A(\bar{w}, \bar{P}) = f^B(\bar{w}, \bar{P}) = 1$ and probabilities to buy the product are given by the following expressions:

$$\begin{cases} q^A &= (1 - P)w \\ q^B &= (1 - P)(1 - w) \end{cases}$$

For a given price this system describes a preference frontier, depicted at Figure 1, which encompasses all admissible pairs of probabilities for two types of consumers to buy the product. By choosing the product design (characteristic of the product) the monopolist identifies probability pair (q^A, q^B) and fixes the network of potential buyers. The network of potential buyers consists of all consumers that would buy the product if they know about it. An increase in P shifts the frontier inwards, simultaneously decreasing probabilities to buy the product for both types.

In the paper we will encounter two special types of the product design.

Definition 1. *The design is called symmetric if the product characteristic w is such that two types of consumers acquire the product with identical probabilities, $q^A = q^B$.*

In the case described above, the symmetric design is represented by $w = \frac{1}{2}$.

Definition 2. *The design is called specialized if the product characteristic $w \in \{0, 1\}$, which implies that only one type of consumers acquires the product.*

These two types of design represent different marketing strategies. A symmetric design intends to satisfy needs of both types of consumers, without giving preference to any of them, while the specialized one focuses on one type and neglects the other.

2.2 Network Structure

The network is represented by a random graph characterized by a degree distribution $p(k)$ and probabilities of ties among types of consumers (ρ^A, ρ^B). The degree distribution describes overall connectivity in the network by assigning probability $p(k)$ for a randomly chosen node to have k contacts. Given the degree distribution, parameters (ρ^A, ρ^B) describe who is connected to whom. Namely, the parameter ρ^i is the probability that a randomly chosen link of a consumer of type i leads to a consumer of the same type and with complementary probability to consumer of another type. The probability that a consumer of type i with k links has $j \leq k$ links to consumers of the same type is thus given by the following binomial expression:

$$Pr\left(J = j|k, \rho^i\right) = \frac{k!}{j!(k-j)!}(\rho^i)^j(1-\rho^i)^{k-j} \tag{1}$$

The expected number of links connecting a type i consumer with k links to consumers of the same type is given by:

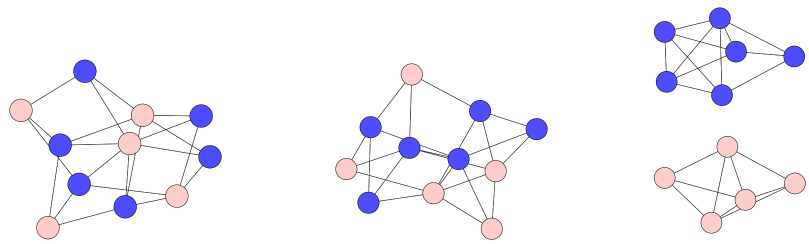
$$\mathbb{E}\left(J|k, \rho^i\right) = \sum_{j=0}^{k}\left[j \times Pr\left(J = j|k, \rho^i\right)\right] = k\rho^i$$

Taking the expectation we find that on average consumer of type A has $z_1(1-\rho^A)$ links to consumers of type B, where z_1 is the expected number of first neighbors (consumers with whom she has direct links). Multiplying the obtained expression by the measure of consumers of type A in the population we obtain that the total number of links of type AB is $\gamma z_1(1-\rho^A)$. By analogy, a number of links of type BA is equal to $(1-\gamma)z_1(1-\rho^B)$. Using the fact that the network is undirected and the number of links of type AB should be equal to the number of links of type BA we obtain the equality $\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$, which we can solve for ρ^B:

$$\rho^B = 1 - \frac{\gamma}{1-\gamma}(1-\rho^A) \tag{2}$$

Therefore, without loss of generality, in the case of two types of consumers we have just one parameter $\rho = \rho^A$ that characterizes linking preferences of all consumers. The parameter ρ represents the level of homophily of the society, since it specifies the probability of friendship relationships among consumers of the same type, for both types[9].

[9]It is important to underline that friendship relationships among consumers are formed on the basis of many parameters such as geographical proximity, common interests and so on. A network formation



(a) (b) (c)

Figure 2: All three graphs have nodes with the same number of neighbors, however they differ in the homophily level. In (a) consumers are linked only to consumers of another type, $\rho = 0$; in (b) we have random mixing of consumers, $\rho = 0.5$; in (c) consumers exhibit extreme homophily, $\rho = 1$.

Figure 2 illustrates 3 different networks with the same degree distribution, moreover all nodes preserve the same connectivity. The only parameter that changes is the level of homophily of the society. As one can observe, depending on ρ networks range from perfectly mixed to two separated graphs, where consumers of type A are completely disjoint from consumers of type B. This suggests that spreading of WOM about different types of products may varies due to the difference in homophily levels that the society exhibits towards these types of products.

To avoid an ambiguity we introduce some key definitions concerning a measurement of homophily level. A benchmark case that we will use extensively is the case when links among consumers are formed with a uniform probability independently of their types.

Definition 3. *The friendship ties in the society are randomly matched if* $\rho = \gamma$.

We can think about the network of detergent consumers as an example of a network with random matching, where preferences towards one type of detergent or another are not correlated. A plausible assumption would be that detergent type that a consumer is using is not important for forming ties with other consumers.

In the sociological literature, a tendency of friendship to be biased towards own type beyond the relative proportion in the population is referred to as "inbreeding homophily" (for survey see McPherson et al., 2001). In this case the proportion of links going to consumers of the same type is higher than otherwise would be implied by random matching.

Definition 4. *The society exhibits the inbreeding homophily if* $\rho > \gamma$

There are also networks in which a situation may be reversed and social ties are biased towards different-type relationships (e.g. network of sexual contacts).

process itself is beyond the scope of this paper. In the analysis we assume that the network of social contacts is exogenously given, and is the same for all products in question.

Definition 5. *The society exhibits a heterophily if* $\rho < \gamma$

To illustrate the idea let us consider examples of networks with inbreeding homophily and random matching. If consumers are matched randomly with a uniform probability then consumer of type A has on average the proportion $\rho = \gamma$ of neighbors of the same type. At the same time expression (2) implies that the average proportion of neighbors of consumer of type B of the same type is $\rho^B = (1-\gamma)$, which equals to the proportion of consumers of type B in the population. In the case when consumers of type A are linked more often among themselves as compared to the case of random matching, $\rho > \gamma$, by expression (2) the same applies to consumers of type B, since $\rho^B > (1-\gamma)$.

2.3 Monopoly Problem

The monopolist develops a new product and introduces it to consumers who are engaged in WOM communication. In the model the monopolist chooses design of the product w and price P to maximize profits. To induce sales the monopolist advertises the product directly to an infinitesimal part of the population selected at random. In the further analysis we relax assumption of random initial sample and allow the monopolist to target consumers by their type. The rest of the population is expected to find out about the product from their neighbors who have acquired the product. The diffusion of information stops when there are no new acquisitions of the product.

We focus on the case of infinitesimal initial sample of consumers who receive advertisement due to two reasons. The first reason is that in this case it is easy to estimate number of purchases resulting from one advertisement. And the second reason is purely technical and comes from limitations of generating functions approach which requires that consumer can get information only from one source.

The network literature usually distinguishes two possible scenarios of information spreading. In the first, information propagates to some finite number of consumers and than stops, while in the second information continues to propagate unboundedly. Let us give the precise definition of the latter case:

Definition 6. *We say that the global cascade of sales arises if ultimately some non-infinitesimal fraction of the population buys the product.*

Depending on wether the global cascade of sales arises or not there are two techniques available to study information diffusion. The main results of the paper are developed for the case of finite sales, while in Section 8 we study the case of global cascade.

3 Cascade of Sales

In this section we derive an expression for the expected size of cascade of sales generated by one advertisement and study its properties. In the derivation of the expression we rely on the generating functions approach for multi-type nodes based on Newman (2003). The main focus of Newman's paper is heterogeneity of types in terms of degree distribution. Our paper adopts different perspective. While two types of consumers have the same degree distribution, they differ in their willingness to purchase the product. This implies that the further propagation of information depends on the way in which different types are interlinked.

3.1 Generating Functions Approach

In the field of complex networks, generating functions were introduced by Newman et al. (2001) and since then have been widely used. A generating function encapsulates all information about degree distribution, and thus completely characterizes a random network. The generating functions allow us to calculate various local and global properties, such as average degree, average size of component, etc.

In the case of nodes of two types we need to define generating functions associated with degree distribution and homophily level of each type of consumer. Recall that probability of having j links to consumers of the same type for a randomly selected consumer of type i with k links is given by $Pr(J = j|k, \rho^i)$, which is described in (1). The probability pseudo-generating function $F_0^i(x, y)$, where $i \in \{A, B\}$, is given by:

$$F_0^i(x, y) = \sum_{k=0}^{\infty} p(k) q^i \sum_{j=0}^{k} Pr(J = j|k, \rho^i) x^j y^{k-j} \tag{3}$$

This is a polynomial expression in x and y where the coefficient of $x^j y^{k-j}$ is the probability that a randomly chosen consumer of type i buys the product, given that she has j links to consumers of the same type and $k - j$ links to consumers of another type. These functions are known as pseudo-generating due to the fact that for $x = y = 1$ they do not sum to 1. This happens since not all consumers buy the product. Actually, $F_0^i(1, 1) = q^i$, which is the probability that a randomly chosen consumer of type i buys the product given that she is aware of it.

Using the binomial identity we can perform summation over j and the expression reduces to the following:

$$F_0^i(x, y) = \sum_{k=0}^{\infty} p(k) q^i [\rho^i x + (1 - \rho^i) y]^k$$

The degree distribution of a neighbor of a randomly chosen consumer plays an important role in the analysis to come. Note, it is not the same as the degree distribution of a randomly selected consumer, since the more links consumer has the more probably she will be encountered as a neighbor. A consumer with k links has k-times higher probability to be selected as a neighbor of a randomly chosen consumer than a consumer with one link. Therefore, the probability to have a neighbor with k links is proportional to $kp(k)$.

After normalization we obtain that the degree distribution of a neighboring consumer $\xi(k)$ is given by:

$$\xi(k) = \frac{kp(k)}{\sum_{j=1}^{\infty} jp(j)} = \frac{kp(k)}{z_1},$$

where z_1 is the average degree of a randomly chosen node. Using the degree distribution of neighboring consumer we can write a generating function characterizing degree distribution of a neighboring node of consumer of type i:

$$F_1^i(x,y) = \sum_{k=0}^{\infty} \xi(k) q^i [\rho^i x + (1-\rho^i) y]^k$$

The important characteristic that affects the process of information diffusion in the network is the excess degree of a neighboring node. The generating functions that characterize the probability that a neighboring consumer of type i has k links apart of the link which led us to this consumer is given by:

$$\hat{F}_1^i(x,y) \quad = \quad \sum_{k=1}^{\infty} \xi(k) q^i [\rho^i x + (1-\rho^i) y]^{k-1}$$

Now we are prepared to formulate an expression that characterizes the size of the cascade of sales generated by one advertisement. The result is summarized in Lemma 1.

Lemma 1. *The expected number of consumers who buy the product if the monopolist advertises it to a randomly chosen consumer is given by an expression:*

$$s(q^A, q^B, \rho, \gamma, z_1, z_2) = (\gamma \ 1-\gamma) \left[\mathbf{I} + \mathbf{F_0'} (\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}')^{-1} \right] \begin{pmatrix} q^A \\ q^B \end{pmatrix},$$

where z_1 and z_2 are expected numbers of first and second neighbors respectively, $\rho^A = \rho$, $\rho^B = 1 - \frac{\gamma}{1-\gamma}(1-\rho)$, $\hat{\mathbf{F}}_\mathbf{0}' = \frac{z_1^2}{z_2} \hat{\mathbf{F}}_\mathbf{1}'$ and

$$\hat{\mathbf{F}}_\mathbf{1}' = \frac{z_2}{z_1} \begin{pmatrix} q^A \rho^A & q^A (1-\rho^A) \\ q^B (1-\rho^B) & q^B \rho^B \end{pmatrix}$$

Proof See appendix □

The first term of the expression $(\gamma \ 1-\gamma) \mathbf{I} \binom{q^A}{q^B}$ is the probability that a randomly chosen consumer buys the product and transmits information further. The second term consists of two parts. The first part $(\gamma \ 1-\gamma) \mathbf{F_0'}$ is a vector with components showing the number of first neighbors of type A and type B consumers who buy the product. The second part is vector $(\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}')^{-1} \binom{q^A}{q^B}$ with components that represent number of purchases generated by the flow of information through a randomly chosen link to consumers of type A and B.

Similar to the epidemic diffusion literature only first two moments of degree distribution are relevant for the propagation of cascade of sales. This substantially reduces the amount of information about network structure that monopolist needs to know to make the optimal decision.

In a special case when consumers of both types have the same preferences towards the product and buy it with the same probability q the expression of cascade of sales reduces the expression of the average size of component of operational nodes obtained in Callaway et al. (2000):

$$s(q^A, q^B, \rho, \gamma, z_1, z_2)|_{q^A=q^B=q} = q + \frac{q^2 z_1}{1 - q(z_2/z_1)}$$

In this case the size of sales cascade is independent of such network characteristics as population composition γ and homophily level ρ. Thus for homophily to have an impact on the information diffusion there should be the heterogeneity of types in terms of preferences towards the product.

4 Main Results

We begin our analysis by establishing a condition under which the global cascade of sales arises. With this condition in mind, we turn to the problem of the monopolist considering the case when sales are finite. We derive the optimal pricing and product design by solving a maximization problem in two steps. In the first step we fix the price and solve the problem for the optimal design of the product. In the second step we allow the monopolist to re-optimize with respect to the price. We complete our analysis by studying implications of the homophily level for the price elasticity of demand and social welfare.

4.1 Arise of the Global Cascade of Sales

A WOM marketing campaign is regarded as successful if it induces multiple sales per advertisement. However, there are some exceptional cases when information propagates to a significant part of the population. These were the case of movie advertisement "The Blair Witch Project" and diffusion of Hotmail accounts. In this section we identify a condition under which the monopolist acting optimally can sell the product to non-infinitesimal part of the population. We consider two cases, when the price is endogenous and forms part of the decision process of the monopolist and when price is exogenous.

From Lemma 1 we know that number of buyers of the product explodes when the denominator $\det(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})$ goes to 0. Thus the condition of appearance of the global cascade of sales is:

$$1 - q^A q^B \left(\frac{z_2}{z_1}\right)^2 (1 - \rho^A - \rho^B) - \frac{z_2}{z_1}(q^A \rho^A + q^B \rho^B) \leq 0$$

In order not to favor any group of consumers in the following analysis we assume that consumers are partitioned into two groups of equal sizes, thus consumers of type A and B constitute half of the population. In this case the expression (2) implies that $\rho^A = \rho^B = \rho$. Substituting expressions for q^A and q^B and incorporating assumptions we obtain the following quadratic inequality:

$$w^2(1-P)^2 \left(\frac{z_2}{z_1}\right)^2 (1-2\rho) - w(1-P)^2 \left(\frac{z_2}{z_1}\right)^2 (1-2\rho) + 1 - (1-P)\frac{z_2}{z_1}\rho \leq 0$$

In this expression, the degree distribution is summarized by the ratio of expected number of second neighbors to expected number of first neighbors. This ratio tells us how efficient is a network in information diffusion. In particular, it shows how many second neighbors on average become aware of the product if a consumer shares the information with one of her first neighbors.

The following proposition summarizes the result:

Proposition 1. *For endogenous price P, if $z_2/z_1 \geq \min\{2, \rho^{-1}\}$ there exists non empty set $E(z_2/z_1, \rho)$ such that for any $(w, P) \in E(z_2/z_1, \rho)$ a global cascade of sales arises.* □

Proof See appendix □

In the framework of one-type nodes a paper by Molloy and Reed (1995) for the first time derives the condition for appearance of the giant component of connected nodes, which is $z_2/z_1 > 1$. In our case it is a necessary condition for a global cascade of sales to occur. One can easily check that $z_2/z_1 < 1$ does not satisfy the condition in Proposition 1, since $\rho \in [0, 1]$. Intuitively, for the information to spread unboundedly, there should exist a giant component of connected consumers upon which spreading may take place.

The condition in Proposition 1 is stronger than $z_2/z_1 > 1$ since not all consumers buy the product and consequently relay WOM about it further. One can separate the condition into two parts: $z_2/z_1 \geq 2$ and $z_2/z_1 \geq \rho^{-1}$. The first part of the condition tells us that independently of homophily level ρ, if z_2/z_1 is higher than 2 then a global cascade of sales occurs. This part of the condition comes from the case when maximal spread of WOM is attained for symmetric characteristic $\left(w = \frac{1}{2}\right)$, which mitigates differences between nodes and makes ρ irrelevant. Moreover, it resembles a condition from Callaway et al. (2000) for the appearance of a giant component of operational nodes, $z_2/z_1 \geq \frac{1}{p}$, where p is a probability that a node is operational. In our model in the case of the symmetric design $p = w = \frac{1}{2}$, since all consumers buy the product with the same probability.

The second part of the condition comes from the case when $\rho > \frac{1}{2}$ and the maximal spread of WOM is attained when the monopolist chooses a specialized design ($w \in \{0, 1\}$).

In this case information propagates only via consumers of one type. To fix ideas assume that the monopolist chooses $w = 1$ and thus only consumers of type A buy the product. The expected number of first neighbors of type A is ρz_1 and the expected number of second neighbors of type A, which can be attained through type A consumers is $\rho^2 z_2$. Substituting these numbers into the condition from Molloy and Reed (1995) we obtain $z_2/z_1 > \rho^{-1}$, which is exactly the same as the second part of the condition in Proposition 1.

In the analysis to come we also consider a case when the price is exogenously given and the monopolist only chooses characteristic of the product w. The following lemma establishes condition for the global cascade of sales to occurin this case:

Lemma 2. *For an exogenously given price P, if $z_2/z_1 \geq \frac{1}{1-P} \min\left\{2, \rho^{-1}\right\}$ there exists non-empty interval $[\underline{w}, \overline{w}]$, such that for any $w \in [\underline{w}, \overline{w}]$ a global cascade of sales arises.*

Proof See appendix □

In the case of an exogenous price the condition of appearance of giant cascade of sales is stricter compared to the case of endogenous price. This happens since not all consumers are prepared to pay a given price P for the product, while in the latter case the monopolist may always set the price equal to zero.

4.2 Optimal Design

In this section we consider the problem of the monopolist who takes the price as given and chooses the design of the product to maximize profits. Without loss of generality we assume that the production cost of the product is zero. Thus profits are given by the product of the price and the size of sales cascade. In the case of an exogenously given price Lemma 2 implies that there is no global cascade if $z_2/z_1 \leq \frac{1}{1-P} \min\left\{2, \rho^{-1}\right\}$. Thus the monopolist profits maximization problem subject to preferences frontier is the following:

$$\max_{w} P \times s(q^A, q^B, \rho, \frac{1}{2}, z_1, z_2)$$

$$\begin{aligned} \text{s.t.:} \quad & q^A = (1-P)w \\ & q^B = (1-P)(1-w) \end{aligned}$$

Before going to the results we develop some intuition. We already have seen the importance of homophily level in the Lemma 2 . In the following let us assume that the society exhibits heterophily, which implies that nodes of type A are more often connected to nodes of type B. Assume further that a consumer of type A has bought the product. Since majority of her neighbors are of type B, a necessary condition for further spread of the information is attractiveness of the product to consumers of type B. However, once they buy the product, most of their neighbors are of type A and the process reiterates. Thus we can conclude that for a sufficiently low homophily level the optimal product design should be appealing to both groups of consumers. Assume now that homophily level is sufficiently high and consumers of both types have majority of their links to the consumers of the same type. The question whether it will be optimal to focus on consumers of just one type is non-trivial. There are components of consumers of both types and if the monopolist focuses on one type all components of another type will be out of reach. The following proposition summarizes the results:

Proposition 2. *For any exogenously given price P following holds:*

(a) if $\rho = \frac{1}{2}$ function $s(\cdot)$ is horizontal line and all $w \in [0, 1]$ are solutions to the maximization problem.

(b) if $\rho < \frac{1}{2}$ function $s(\cdot)$ is quasi-concave and has its unique maximum at the point $w = \frac{1}{2}$.

(c) if $\rho > \frac{1}{2}$ function $s(\cdot)$ is quasi-convex and its unique minimum is at the point $w = \frac{1}{2}$ and maxima are situated at points $w \in \{0, 1\}$

Proof See appendix □

The first result states that for two groups of consumers of equal sizes if $\rho = \frac{1}{2}$ (which for $\gamma = \frac{1}{2}$ implies random mixing) the size of sales cascade is not affected by the product characteristic w. That is why heterogeneity of consumers preferences towards the product and towards linking both constitute key ingredients of the model.

The Proposition 2 confirms our intuition for the case of low homophily levels and most importantly states that the maximization problem has a threshold solution. More precisely, independently of a degree distribution and price, if ρ becomes higher than $\frac{1}{2}$, the optimal product design abruptly changes from the symmetric $w^* = \frac{1}{2}$ to specialized $w^* \in \{0, 1\}$. The explanation is the following: when ρ is higher than $\frac{1}{2}$ the majority of consumer's neighbors are of the same type as the consumer. That is why the design most attractive for a randomly selected consumer is the one that induces the highest sales of the product among her neighbors. The situation reiterates for every consumer that buys the product, reinforcing the optimality of the specialized design.

The obtained result does not depend on a degree distribution or price, which makes it easy to apply. All the information the monopolist needs to know is whether homophily level of the society towards the product is higher or lower than $\frac{1}{2}$ to choose the optimal design.

4.3 Demand

Incorporating the optimal design of the product into the expression for cascade of sales from Lemma 1 we obtain the following demand function:

$$Q(P,\rho,z_1,z_2) = \begin{cases} \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{2-z_2/z_1(1-P)}\right), & \rho \le \frac{1}{2} \\ \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{\frac{1}{\rho}-z_2/z_1(1-P)}\right), & \rho > \frac{1}{2} \end{cases}$$

Note that for $\rho \le \frac{1}{2}$ the demand is independent of homophily level ρ, since in this case the optimal design is given by the symmetric characteristic $w^* = \frac{1}{2}$. The symmetric design implies that both types of consumers buy the product with the same probability and mixing pattern does not matter. The following proposition summarizes main properties of the demand function:

Proposition 3. *The demand function $Q(P,\rho,z_1,z_2)$ has the following properties:*

(i) $Q(P,\rho,z_1,z_2)$ for $\rho > \frac{1}{2}$ is continuous, increasing and convex in ρ.

(ii) $Q(P,\rho,z_1,z_2)$ is decreasing and convex in P.

(iii) The price elasticity of demand is increasing in ρ, for $\rho > \frac{1}{2}$.

Proof See appendix □

The first result states that for homophily level $\rho > \frac{1}{2}$ a classical demand (the demand with the incorporated optimal design) increases in ρ. In Proposition 2 we have seen that for $\rho > \frac{1}{2}$ the optimal design is specialized with characteristic w^* belonging to the set $\{0, 1\}$. In this case a randomly chosen consumer has the majority of neighbors of the same type and a further increase of homophily increases this subset. To fix ideas, assume that $w^* = 1$ and $P = 0$. Thus if a consumer of type A gets information about the product she and all her neighbors of type A acquire the product. Thus an increase in the homophily level leads to a higher number of acquisitions in the neighborhood of type A consumer.

The obtained result differs from intuitions of McPherson (2001), which argues that for higher homophily levels, information flows are localized and status quo of individuals tends to be maintained. We show that when further transmission of information depends on the adoption decision, an increase in the homophily may actually produce higher spread of information. Higher levels of homophily induce a higher correlation of consumers' preferences making it easier for the monopolist to design a product information of which can penetrate further in the network.

The convexity part of the result *(i)* comes from the fact that an increase of homophily expands a subset of neighbors of the same type in a consumer's neighborhood. Moreover, each consequent increase of homophily operates upon the neighborhood of a higher number of consumers, which acquire the product. This generates increasing returns of the number of buyers in homophily level.

The result *(ii)* has a similar nature as the result of convexity of demand in ρ. A price increase affects the decision of all consumers to acquire the product, regardless of their position in the chain of buyers. Let us consider an example when a consumer who is situated earlier in the chain of buyers stops to acquire the product due to a price increase. In this case the whole branch of consumers that receives information about the product through this consumer stops to acquire the product. A further price increase has smaller impact on the demand, since chains of buyers become shorter.

Having two previous results at hand we are equipped to understand the third one. The result *(i)* implies that when $\rho > \frac{1}{2}$ an increase in ρ leads to higher sales and awareness of consumers about the product. Hence, a price increase affects decision of an increased number of consumers, which translates into an increased price elasticity of the demand.

4.4 Optimal Price

In previous analysis we have seen that the optimal product design is independent of the price chosen by the monopolist, and thus all results derived previously hold for the optimal price as well. The monopolist maximizes profits and solves the following problem with respect to the price:

$$\max_{0 \leq P \leq 1} P \times \frac{1-P}{2} \left(1 + \frac{z_1(1-P)}{\frac{1}{\rho} - \frac{z_2}{z_1}(1-P)} \right)$$

In a price setting the monopolist faces usual trade-off: an increase in the price augments profits from each unit sold, but simultaneously decreases demand for the product. However, in the presence of WOM communication there is an additional informational component of the trade-off. Since consumers spread information about the product only if they acquire it, a price increase lowers product awareness of consumers. In the case of full information there are no information spreading considerations and the optimal price is $P^*_{FU} = \frac{1}{2}$. The following proposition summarizes properties of the optimal pricing strategy:

Proposition 4. *The following holds:*

(a) The optimal price P^* is decreasing in the homophily level for $\rho \geq \frac{1}{2}$, while for $\rho < \frac{1}{2}$, P^* is independent of the homophily level.

(b) The optimal price P^* is always lower than $P^*_{FU} = \frac{1}{2}$.

(c) For two degree distributions $p(k)$ and $p'(k)$ and corresponding optimal prices P^* and P'^* if $p(k)$ is the mean preserving spread of $p'(k)$ then $P^* < P'^*$.

Proof See appendix □

Part (i) of the result is a direct consequence of the fact that the price elasticity of demand is increasing in the homophily level. As we have seen in Proposition 3 for $\rho > \frac{1}{2}$ an increase of homophily level implies that more consumers become aware about the product, and sales increase. The monopolist by reducing the price capture a higher fraction of informed consumers. Proposition 4 implies that the informational component outweighs an increase of profits per purchase generated by a higher price.

4.5 Social Welfare

The majority of literature on homophily adopts a view that the diversity of contacts is a socially desirable property per se (e.g. see Moody, 2001). However, a recent paper by Currarini et al. (2009) shows that welfare implications of homophily crucially depend on the structure of consumers preferences. Our model allows us to address welfare implications of the homophily in an explicit manner.

A consumer surplus is given by the following expression:

$$CS(P^*(\rho), \rho, z_1, z_2) = \int_{P^*(\rho)}^{1} Q(P, \rho, z_1, z_2) dP$$

As we know by the Proposition 3 for $\rho \geq \frac{1}{2}$ an increase in the homophily level shifts the demand curve upwards. This happens, since more consumers become aware of the product. At the same time an increase in the homophily level by Proposition 4 leads to a lower price. As a consequence more consumers buy the product for lower price and thus both effects lead to an increase of consumer surplus.

In the case of $MC = 0$, a producer surplus is the following expression:

$$PS(P^*(\rho), \rho, z_1, z_2) = P^*(\rho) \times Q(P^*(\rho), \rho, z_1, z_2)$$

The producer surplus is increasing function in the homophily level. This is easy to see if we fix the price and increase ρ. Since demand is increasing in ρ the function is increasing too. If we relax assumption about exogenous price and let the monopolist to re-optimize, the producer surplus will increase even further.

Proposition 5. *Both consumer surplus and monopolistic profits are increasing in the level of homophily.* □

The Proposition 5 states that if information retransmission is subject to an adoption decision then the society is better-off when homophily level is high. There are two driving forces of the result. First, the optimally constructed message propagates better in homogenous groups, which leads to an increase in awareness of consumers about the product. Second, the price reduction is more effective in facilitating diffusion of WOM in the case of higher homophily levels. These two effects are beneficial for both consumers and the monopolist.

5 The Effect of Network Density

In further analysis we will often refer to a special case of a network structure known as a classical random graph. This notion has been introduced by Paul Erdõs and Alfréd Rényi (1959) and since then it is the most studied model of network. Node's connectivity in a random graph follows the Poisson degree distribution and arises in infinite network, where each node creates a link to any other node in the network at a uniform probability.

In the case of Poisson degree distribution the average connectivity z_1 is a sufficient characteristic of the network. The expected number of second neighbors z_2 in this case equals to z_1^2. This property of classical random graph allows us to address the effect of network density on the propagation of WOM and the optimal strategies of the monopolist.

In our model the probability that a randomly selected link connects two consumers of the same type is different from the probability that it connects consumers of different types. One can think about a network of N consumers of two types, where each consumer creates a link to any other consumer of the same type with probability $\frac{\rho z_1}{N}$ and to a consumer of another type with the probability $\frac{(1-\rho)z_1}{N}$. When N goes to infinity we obtain infinite network that is characterized by two Poisson degree distributions. One for links among consumers of the same type with mean ρz_1 and another for links among consumers of different types with mean $(1-\rho)z_1$. Recall, that the sum of two Poisson variables also follows Poisson distribution with the mean equal to the sum of means. That is why the overall connectivity of a randomly chosen node follows Poisson distribution and our network is a classical random graph with average connectivity given by z_1.

As we already have seen the optimal design of the product does not depend on the degree distribution and is given by the expression in Proposition 2. Incorporating relation between the expected number of first and second neighbors into the demand function we obtain:

$$Q(P,\rho,z_1,z_2) = \begin{cases} \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{2-z_1(1-P)}\right), & \rho \leq \frac{1}{2} \\ \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{\frac{1}{\rho}-z_1(1-P)}\right), & \rho > \frac{1}{2} \end{cases}$$

In the further derivations of this section we will consider the case of $\rho > \frac{1}{2}$, since otherwise homophily level does not affect spreading. Taking derivative with respect to z_1 of the demand function one can show that the denser is the network the higher is the demand:

$$\frac{\partial}{\partial z_1}Q(P,\rho,z_1,z_2) = \frac{(1-P)^2\rho}{2(1-(1-P)z_1\rho)^2} > 0$$

Solving maximization problem we obtain the expression for the optimal price,[10,11] which is given by:

$$P^* = 1 - \frac{1 - \sqrt{1 - z_1\rho}}{z_1\rho}$$

For $\rho > \frac{1}{2}$ the derivative of the optimal price with respect to homophily level is negative, thus the optimal price is decreasing in the homophily level:

$$\frac{\partial P^*}{\partial \rho} = -\frac{2 - z_1\rho - 2\sqrt{1 - z_1\rho}}{2z_1\rho^2\sqrt{1 - z_1\rho}} < 0$$

To study the effect of the network density on the optimal price we derive the expression (5) with respect to z_1:

$$\frac{\partial P^*}{\partial z_1} = -\frac{2 - z_1\rho - 2\sqrt{1 - z_1\rho}}{2z_1^2\rho\sqrt{1 - z_1\rho}} < 0$$

The derivative is negative, which implies that the optimal price P^* is decreasing in both average connectivity and homophily parameter ρ.

Concerning the welfare implications of homophily, using the same line of arguments as we have outlined before one can show that an increase in z_1 leads to a higher consumer surplus. This happens since a denser network implies higher awareness of consumers about the product. An increase in the network density also augments benefits for the monopolist of price reduction. These both effects are beneficial for consumers.

6 Targeted Advertisement

In the previous analysis we have considered the problem of the monopolist, which cannot distinguish consumers by type. The monopolist, restricted by an anonymity assumption, was advertising the product to a randomly chosen subset of the population. This formulation is relevant for an advertisement through the mass media, when the monopolist cannot control who is watching or hearing the advertisement. However, in the case of a direct advertisement there is a possibility to target chosen group of consumers.

In this section we are going to relax anonymity assumption and allow the monopolist to observe types of consumers. More precisely, we assume that the monopolist chooses a design of the product w and the proportion α of consumers of type A that will receive the advertisement. For the tractability of the problem, we assume that price P is exogenously given. Thus the maximization problem of the monopolist becomes:

[10] In the case of Poisson degree distribution the first order condition for the optimal price when $\rho > \frac{1}{2}$ reduces to $z_1\rho P^2 + 2P(1 - z_1\rho) + z_1\rho - 1 = 0$.

[11] Condition of absence of a global cascade of sales in the case of the Poisson distribution is $z_1 < \min\{2, \rho^{-1}\}$.

$$\max_{w,\alpha} P \times s(q^A, q^B, \rho, \alpha, z_1, z_2)$$

$$\text{s.t.:} \quad q^A = (1-P)w$$
$$q^B = (1-P)(1-w)$$

The expression for sales cascade $s(q^A, q^B, \rho, \alpha, z_1, z_2)$ can be rewritten as a linear combination of number of purchases resulted from the advertisement of the product to a consumer of type A and of type B: $\alpha \times s^A(q^A, q^B, \rho, z_1, z_2) + (1-\alpha) \times s^B(q^A, q^B, \rho, z_1, z_2)$. Given the linear structure of the problem in terms of α it is easy to see that if $q^A \neq q^B$ the optimal targeting proportion has a corner solution. Namely, the solution depends whether a cascade of sales triggered by an advertisement is higher if advertisement receives consumer of type A or of type B. In the case when $q^A = q^B$ both types of consumers buy the product with the same probability and thus all values of α in the interval $[0, 1]$ are optimal.

Proposition 6. *Targeting one type of consumers for advertisement is always the optimal strategy for the monopolist.* □

Since the preference frontier is symmetric, without loss of generality we assume that the monopolist targets consumers of type A and hence $\alpha^* = 1$. Thus if $\alpha^* = 1$ and some w^* is a solution of the problem then $\alpha^* = 0$ and $1 - w^*$ is a solution too.

The problem with targeted advertisement for an arbitrary degree distribution quickly becomes analytically non-tractable. The following proposition summarizes the results obtained for a commonly employed network structure - classical random graph.

Proposition 7. *In the case of the Poisson degree distribution and exogenously given price, there is a threshold level* $\hat{\rho}_T(z_1, P) = \frac{3}{4} + \frac{1-\sqrt{9-2z_1(1-P)+(1-P)^2 z_1^2}}{4(1-P)z_1} \leq \frac{1}{2}$, *such that for* $\rho \geq \hat{\rho}_T(z_1, P)$ *the monopolist will only advertise to and specialize on one type of consumers* ($\alpha^* = 1$ *and* $w^* = 1$)*. For* $\rho < \hat{\rho}_T(z_1, P)$ *the optimal advertisement strategy is* $\alpha^* = 1$, *while the optimal characteristic is given by the following expression:*

$$w^* = \rho - \frac{1 - 2\rho - \sqrt{(1-\rho)(1-2\rho)(1-(1-P)z_1\rho)[2+(1-P)z_1(1-2\rho)]}}{(1-P)z_1(1-2\rho)} > \frac{1}{2}$$

Proof See appendix □

Proposition 7 shows that the possibility of targeting advertisement inevitably brings bias. The bias has two forms. The first one is that the threshold level of homophily that separates specialized and symmetric designs moves to a level lower than $\frac{1}{2}$. The second form is that for ρ lower than the new threshold level, the optimal design belongs to the interval $\left(\frac{1}{2}, 1\right)$ instead of being symmetric as before. This occurs since the success of

WOM campaign to a high extend depends on the effectiveness of the direct advertisement in inducing initial acquisitions of the product. In order to persuade consumers of type A, who receive direct advertisement, to buy the product the monopolist designs it more attractive to them. The bias in the product design persists even when the society exhibits heterophily ($\rho < \frac{1}{2}$).

In the case of the random graph connectivity z_1 is the only parameter that governs the network. When we approach the global cascade phase as z_1 goes to $2(1-P)^{-1}$, threshold level $\hat{\rho}_T(z_1, P)$ approaches $\frac{1}{2}$ and the optimal characteristic for $\rho < \frac{1}{2}$ becomes $w^* = \frac{1}{2}$. The optimal design becomes exactly the same as in the case of non-targeted advertisement. Thus the monopolist optimally sacrifices some fraction of initial adopters, by designing the product in a such way that WOM can penetrate further in the network.

This result implies that when we approach the global cascade phase the option to target some group of consumers does not alter neither the demand nor the optimal design of the product. We regard this finding as the indication of robustness of the results obtained in the base-line model.

7 Targeting One Type of Consumers

In this section we address the problem of the monopolist who has interest only in one type of consumers, for concreteness lets assume that this is type B. This situation may arise if the monopolist believes that consumers differ in their post purchasing behavior. For example, once a consumer of type B buys the product she becomes a loyal customer and continues to purchase products of the same brand, while consumers of type A are accidental buyers. For the sake of simplicity we assume that the monopolist completely ignores consumers of type A. Assume further that the monopolist maximizes awareness of the brand and chooses price equal to 0. The main question is than: what is the optimal product design that maximizes a number of purchases by consumers of type B?

The first guess could be that the monopolist should completely forget about consumers of type A and design the product as attractive as possible to consumers of type B. The first guess, however, turns out to be wrong for broad set of parameters. Assume for example that homophily level of the society is low, which implies that consumers of type B are mostly connected to consumers of type A. Hence, to spread, information should be able to pass through consumers of type A. A Figure 3a illustrates the optimal product design for consumer groups of equal sizes ($\gamma = \frac{1}{2}$), $z_1 = 1.7$ and $z_2 = 2$. Note that for $\rho \in [0, 0.39]$ the optimal design is such that there is a non-zero probability for consumers of type A to buy the product. The result, however, requires low levels of homophily and actually implies heterophily of the society. We already have seen similar picture in our base-line model when the monopolist profits from both types of consumers.

Probably, the more surprising result is that although society exhibits homophily it may be optimal to make a product attractive for consumers of type A. The only requirement

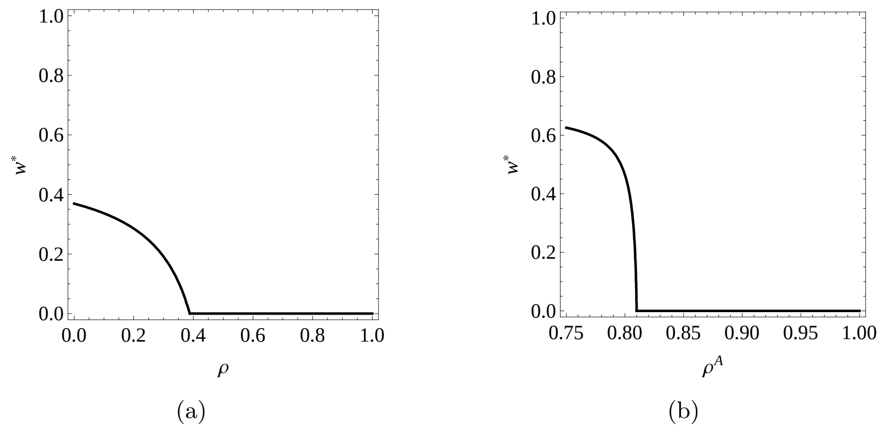


Figure 3: The optimal design when $z_1 = 1.7$, $z_2 = 2$. Figure (a) for the case of $\gamma = \frac{1}{2}$, figure (b) for the case of $\gamma = 0.8$

is that the proportion of consumers of type A should be sufficiently high. A Figure 3b illustrates the optimal product design for the case when consumers of type A constitute 80% of the population ($\gamma = 0.8$) and the expected number of neighbors are as before: $z_1 = 1.7$ and $z_2 = 2$. Note that $\rho \in [0.8, 1]$ implies that the society exhibits homophily and there is a range of $\rho \in [0.8, 0.81]$ such that the optimal product characteristic w is non-zero. Another surprising feature of the result is that for a sufficiently small ρ it is optimal to make the product more attractive to consumers of type A than of type B.

8 Robustness check

In this section we address the issue of robustness of our model. We start our analysis by relaxing the assumption of linearity of preference frontier. The preference frontier has non-linear shape when a change in the characteristic affects attractiveness of the product asymmetrically for two types of consumers. For example, speeding up smartphone may make teenagers very happy, since they can play their favorite games, but may make unhappy consumers who value longer battery life of their phones. In such situations the shape of the frontier can vary from the concave to convex, depending on a product and characteristic in question.

In previous sections we have seen what happens when WOM marketing campaign does not trigger a global cascade of sales. This is the case for a majority of WOM campaigns. However, WOM campaigns such as diffusion of Hotmail accounts and “The Blair Witch Project” (tiny budget movie) were so successful that a considerable fraction of the population became aware of the product. In these cases we can no longer apply techniques from the previous analysis.

8.1 Non-Linear Shape of the Preference Frontier

In this section we want to address the robustness of obtained results to a change in the curvature of the preference frontier that the monopolist faces. We consider CES functional form of the preference frontier, which allows to model variety of shapes. Thus probabilities to buy the product for two types of consumers are related in the following manner:

$$\left(q^A\right)^r + \left(q^B\right)^r = (1-P)^r$$

By varying the parameter r we can obtain shapes of the preferences frontier that include a bend inward circle ($r = 0.562$), a linear function ($r = 1$), a bend outward circle ($r = 2$) and everything in between.

Similarly to the analysis in Section 5 we assume that the degree distribution is Poisson and the price is given exogenously. In this case we identify homophily level such that symmetric and specialized designs are solutions. The results are summarized by the following proposition:

Proposition 8. *For an exogenous price and Poisson degree distribution following holds:*

(a) For $r \leq 1$ there is $\hat{\rho}_{NL}(r, P, z_1) = \frac{1}{2} - \frac{2^{\frac{1}{r}}-2}{2z_1(1-P)}$ such that the optimal design is symmetric $w^* = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ if $\rho < \hat{\rho}_{NL}(r, P, z_1)$ and otherwise the optimal design is specialized $w^* \in \{0, 1\}$.

(b) For $r > 1$ the optimal design is symmetric $w^* = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ if $\rho < \hat{\rho}_{NL}(r, P, z_1)$ otherwise the optimal design belongs to the interval $\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1\right)$.

Proof See appendix □

The Proposition 8 states that the optimal design has similar structure as in the case of linear preferences frontier. More precisely, for bend inwards frontier ($r \leq 1$) only symmetric and specialized designs are optimal. They are separated by new threshold value $\hat{\rho}_{NL}(r, P, z_1)$. For bend outwards frontier ($r > 1$) and sufficiently low levels of ρ, symmetric design is optimal. However, for high levels of ρ the solution gradually changes from the symmetric to specialized.

To check the robustness of the obtained result to the selection of degree distribution we consider a numerical solution of the problem for the case of a scale-free degree distribution. Figure 4 shows a diagram of the structure of the solution. One can see that for $r < 1$ we have similar results as in the linear case. Namely, there is the threshold level $\hat{\rho}_{NL}(r, z_1, z_2, P)$ such that for $\rho < \hat{\rho}_{NL}(r, z_1, z_2, P)$ the optimal solution is symmetric and for values of $\rho > \hat{\rho}_{NL}(r, z_1, z_2, P)$ the solution is specialized. In the case of $r > 1$ the structure stays the same, but after $\rho = \hat{\rho}_{NL}(r, z_1, z_2, P)$ the solution gradually changes from symmetric to some intermediate value, which lies in the interval $\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1\right)$.

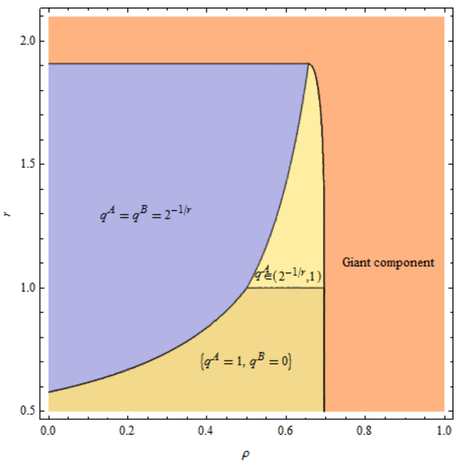


Figure 4: Diagram depicts a solution for the case of scale free distribution with pdf $Ck^{-3.34}$, where C is normalizing constant. In this case $z_1 = 1.23$ and $z_2 = 1.77$. Areas represent: symmetric solution (blue), asymmetric (light yellow), specialized (yellow) and area where a global cascade arises (red).

8.2 Global Cascade Phase

In this subsection we assume that conditions are such that a global cascade of sales arises, by Lemma 2 this happens when $\frac{z_2}{z_1} > \frac{1}{1-P} \min\{2, \rho^{-1}\}$. To determine the fraction of the population that buys the product we turn back to generating functions. However, instead of looking on the distribution of sizes of cascades, we would like to estimate a fractional size of a global cascade.

As in the previous analysis we assume that there are equal proportions of consumers of type A and type B in the population ($\gamma = \frac{1}{2}$). The maximization problem of the monopolist is summarized by the following lemma:

Lemma 3. *For two groups of equal sizes the maximization problem of monopolist becomes:*

$$\max_{q^A, q^B} \frac{1}{2} \left(q^A[1 - G_0(x)] + q^B[1 - G_0(y)] \right)$$

$$\begin{aligned} s.t.: \quad & x = 1 - \rho q^A[1 - \hat{G}_1(x)] - (1-\rho) q^B [1 - \hat{G}_1(y)] \\ & y = 1 - (1-\rho) q^A [1 - \hat{G}_1(x)] - \rho q^B [1 - \hat{G}_1(y)] \\ & 0 \leq q^A, q^B, x, y \leq 1 \end{aligned}$$

where $x = \rho^A u^A + (1 - \rho^A) u^B$ is the probability that a randomly chosen link of a consumer of type A leads to a giant component of buyers and $y = \rho^B u^B + (1 - \rho^B) u^A$ is the same for consumers of type B. In addition $G_0(x) = \sum_{k=0}^{\infty} p(k) x^k$ and $\hat{G}_1(x) = \sum_{k=0}^{\infty} \xi(k) x^{k-1}$. ***Proof*** *See appendix* □

For linear preferences frontier a solution to the maximization problem is characterized by the following proposition:

Proposition 9. *In the case when population is divided into two equally sized groups,* $\gamma = \frac{1}{2}$*, for any degree distribution the following holds:*

- *For* $\rho < \frac{1}{2}$*,* $w = \frac{1}{2}$ *is a local maximum, which gives higher profits than* $w \in \{0, 1\}$*.*
- *For* $\rho > \frac{1}{2}$*,* $\{0, 1\}$ *are local maxima, which give higher profits than* $w = \frac{1}{2}$*.*
- *For* $\rho = \frac{1}{2}$*, the interval* $[0, 1]$ *is the solution to the problem.*

The Proposition 9 indicates that the optimal design of the product has the same structural form as in the case where there is no global cascade of sales.

9 Conclusion

The importance of a word of mouth communication for a company's performance is well documented by a growing amount of literature. However, success of a WOM marketing campaign varies enormously between product categories and within. We show that a high variation in the performance of WOM campaigns can be explained by different homophily levels of a consumer network towards different products.

A key innovation of our paper is two-fold. First, we enrich network structure by incorporating the notion of homophily and study its impact on the optimal strategies of the monopolist. Second, the monopolist is allowed to construct a message to the network by choosing the design of the product. We show that for low levels of homophily the product, designed to attract both types of consumers is preferred to specialized products, even if there is no cost of producing more than one product. The price elasticity of demand is higher for products towards, which consumers network exhibits higher levels of homophily. Finally, we show that social welfare is increasing in homophily level.

Flexibility of the model allows us to outline several avenues for a future research. The first one consists in introduction of influencers, consumers whose opinion affects opinion of many others. In the case when society exhibits homophily, influencers will be linked among themselves and will constitute a core with access to a large share of consumers. The extension is aimed to study the impact of homophily on information spreading in "hub-and-spoke" networks. In the second extension we would like to consider two groups of consumers with different valuation of the product. The main idea here is to study the effect of homophily on the optimal pricing strategy of the monopolist.

References

[1] Buhai, S. and Van der Leij, M., 2008, "A Social Network Analysis of Occupational Segregation," working paper.

[2] Callaway, D. S., Newman M. E. J., Strogatz S. H., and Watts D. J., 2000, "Network robustness and fragility: Percolation on random graphs," Physical Review Letters 85, pp. 54685471.

[3] Campbell, A., 2009, "Tell Your Friends! Word of Mouth and Percolation in Social Networks," job-market paper.

[4] Currarini, S., Jackson, M. O. and Pin, P., 2009, "An economic model of friendship: Homophily, minorities and segregation," Econometrica, Volume 77, Issue 4, pp. 1003-1045.

[5] DiNardo, J., 2006, "Freakonomics: Scholarship in the Service of Storytelling," American Law and Economics Review, Volume 8, Issue 3, pp. 615-626

[6] DiNardo, J., 2007, "Interesting questions in Freakonomics," Journal of Economic Literature, Volume 45, Issue 4, pp. 973-1000

[7] Erdős, P., Rényi, A., 1959, "On Random Graphs. I," Publicationes Mathematicae 6, pp. 290297.

[8] Iribarren, J.L. and Moro, E., 2007, "The network laws of viral marketing," submitted.

[9] Galeotti, A. and Goyal, S., 2008, "A Theory of Strategic Diffusion," forthcoming in Rand Journal of Economics.

[10] Galeotti, A. and Mattozzi, A., 2008, "'Personal Influence': Social Context and Political Competition," submitted.

[11] Geroski P. A., 2000, "Models of technology diffusion," Research Policy Volume 29, Issues 4-5, April 2000, Pages 603-625

[12] Golub, B. and Jackon, M., 2009, "How Homophily Affects Learning and Diffusion in Networks," working paper.

[13] Leskovec, J., Adamic, L. A. and Huberman, B. A., 2007, "The Dynamics of Viral Marketing," In Proc. 7th ACM Conference on Electronic Commerce.

[14] Lopez-Pintado, D. and Watts, D. J., 2008, "Mass Media versus Word of Mouth," working paper.

[15] McPherson, M., Smith-Lovin, L. and Cook, J.M., 2001, "Birds of a Feather: Homophily in Social Networks," Annu. Rev. Sociol., 27, pp. 41544

[16] Moody, J., 2001, "Race, school integration, and friendship segregation in America," American Journal of Sociology, 107 (3), pp. 679716.

[17] Newman, M., 2002, "The Spread of Epidemic Diseases on Networks," Physical Reveiw E, Volume 66(1), art. no. 016128.

[18] Newman, M., 2003, "Mixing patterns in networks," Physical Reveiw E, Volume 67, art. no. 026126.

[19] Reichheld, F.F., 2003, "The one number you need to grow," Harvard Business Review, Vol. 81 No.12, pp.46-54.

[20] Riley, E. and Wigder, Z. D., 2007, "Viral Marketing - Bringing the Message to the Masses," Jupiter Research.

[21] Rubinstein, A., 2006, "Freak-Freakonomics," The Economists' Voice, Volume 3, Issue 9, Article 7.

[22] Sander, L., Warren, C., Sokolov, I., Simon, C., and Koopman, J., 2002, "Percolation on Heterogeneous Networks as a Model for Epidemics," Mathematical Biosciences, Volume 180, Issue 1-2, pp. 293-305.

[23] Valat, E., 2009, "Homophily in Social Networks and Labor Market Outcomes," working paper.

10 APPENDIX

Proof of Lemma 1

Let us find first what is the number of consumers of type A that buy the product if we advertise it to consumer of type A. The answer is:

$$\frac{\partial}{\partial x} H_0^A(x,y)\bigg|_{x=1,y=1} = H_{0x}^A(1,1)$$

With abuse of notation we assume that all function are being evaluated at point $(1,1)$:

$$H_{0x}^A = F_0^A + F_{0x}^A H_{1x}^A + F_{0y}^A H_{1x}^B$$

We can find H_{1x}^i by solving linear system of self-consistency conditions:

$$\begin{cases} H_{1x}^A = \hat{F}_1^A + \hat{F}_{1x}^A H_{1x}^A + \hat{F}_{1y}^A H_{1x}^B \\ H_{1x}^B = \hat{F}_{1x}^B H_{1x}^A + \hat{F}_{1y}^B H_{1x}^B \end{cases}$$

In vector form:

$$\begin{pmatrix} 1-\hat{F}_{1x}^A & -\hat{F}_{1y}^A \\ -\hat{F}_{1x}^B & 1-\hat{F}_{1y}^B \end{pmatrix} \begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

or in more compact way

$$\left(\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}'\right) \begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix},$$

where $\hat{F}_1^i = \sum_{k=1}^{\infty} \xi(k) q_k^i$ and

$$\hat{\mathbf{F}}_\mathbf{1}' = \begin{pmatrix} \hat{F}_{1x}^A & \hat{F}_{1y}^A \\ \hat{F}_{1x}^B & \hat{F}_{1y}^B \end{pmatrix} = \sum_{k=1}^{\infty} \xi(k)(k-1) \begin{pmatrix} q_k^A \rho^A & q_k^A(1-\rho^A) \\ q_k^B(1-\rho^B) & q_k^B \rho^B \end{pmatrix}$$

The number of consumers of type A who buy the product if consumer of type i finds out about the product from one of her friends H_{1x}^i goes to infinity when determinant of the matrix $\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}'$ goes to zero:

$$\Delta = \det \begin{pmatrix} 1-\hat{F}_{1x}^A & -\hat{F}_{1y}^A \\ -\hat{F}_{1x}^B & 1-\hat{F}_{1y}^B \end{pmatrix}$$

The system has following solution:

$$\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = (\mathbf{I} - \hat{\mathbf{F}}_\mathbf{1}')^{-1} \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

Thus we can find the number of consumers of type A who buy the product if consumer of type A receives direct advertisement:

$$H_{0x}^A = F_0^A + (F_{0x}^A \ F_{0y}^A)\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = F_0^A + (F_{0x}^A \ F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

By doing analogous calculations we can get:

$$H_{0x}^B = (F_{0x}^B \ F_{0y}^B)\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = (F_{0x}^B \ F_{0y}^B)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

$$H_{0y}^A = (F_{0x}^A \ F_{0y}^A)\begin{pmatrix} H_{1y}^A \\ H_{1y}^B \end{pmatrix} = (F_{0x}^A \ F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix}$$

$$H_{0y}^B = F_0^B + (F_{0x}^B \ F_{0y}^B)\begin{pmatrix} H_{1y}^A \\ H_{1y}^B \end{pmatrix} = F_0^B + (F_{0x}^B \ F_{0y}^B)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix}$$

The total number of purchases resulted from direct advertisement to a consumer of type A is following:

$$H_{0x}^A + H_{0y}^A = F_0^A + (F_{0x}^A \ F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix} + (F_{0x}^A \ F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix} =$$

$$= F_0^A + (F_{0x}^A \ F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

If the monopolist advertises the product to consumer of type B:

$$H_{0x}^B + H_{0y}^B = F_0^B + (F_{0x}^B \ F_{0y}^B)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

Let us define:

$$F'_0 = \left(\begin{array}{cc} F_{0x}^A & F_{0y}^A \\ F_{0x}^B & F_{0y}^B \end{array} \right) = \sum_{k=1}^{\infty} kp(k) \left(\begin{array}{cc} q_k^A \rho^A & q_k^A(1-\rho^A) \\ q_k^B(1-\rho^B) & q_k^B \rho^B \end{array} \right)$$

The resulting number of purchases resulting from advertisement to consumers of type A and B in vector form is:

$$\mathbf{s} = \begin{pmatrix} H_{0x}^A + H_{0y}^A \\ H_{0x}^B + H_{0y}^B \end{pmatrix} = \begin{pmatrix} F_0^A \\ F_0^B \end{pmatrix} + \mathbf{F'_0}(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

Thus the number of purchases resulting from advertisement to a randomly drawn consumer is:

$$s = (\gamma \; 1-\gamma)\mathbf{s} = (\gamma \; 1-\gamma)\left[\begin{pmatrix} F_0^A \\ F_0^B \end{pmatrix} + \mathbf{F_0'}(\mathbf{I} - \hat{\mathbf{F}}_1')^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}\right]$$

Assuming that the probability to purchase the product does not depend on the number of neighbors that consumer has, namely $q_k^A = q^A$ and $q_k^B = q^B$ we obtain:

$$s = (\gamma \; 1-\gamma)\mathbf{s} = (\gamma \; 1-\gamma)\left[\mathbf{I} + \mathbf{F_0'}(\mathbf{I} - \hat{\mathbf{F}}_1')^{-1}\right]\begin{pmatrix} q^A \\ q^B \end{pmatrix}$$

Note that expression depends on the linear combination of probability to infect initial node $(\gamma \; 1-\gamma)\binom{q^A}{q^B}$ and $(\gamma \; 1-\gamma)\left(\mathbf{I} - \hat{\mathbf{F}}_1'\right)^{-1}\binom{q^A}{q^B}$ which is number of infected nodes if we follow randomly chosen link, with weight given buy $\frac{z_1^2}{z_2}$.

Proof of Proposition 1

The global cascade of sales arises when inequality holds:

$$w^2(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) - w(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) + 1 - (1-P)\frac{z_2}{z_1}\rho \leq 0$$

We want to identify a condition such that there exists characteristic of the product w which satisfies the inequality and hence the global cascade of sales may arise. To this end we find the minimum of the expression and check when it is less than zero. A derivative of the expression with respect to w is $-(1-P)^2(z_2/z_1)^2(1-2\rho)(1-2w)$. Note that if $\rho < \frac{1}{2}$ coefficient of the term w^2 is positive and thus we have upward sloping parabola. In this case function has its minimum at the point $w = \frac{1}{2}$. Substituting to the expression and taking positive root we obtain a condition $z_2/z_1 > 2(1-P)^{-1}$. On the other hand, if $\rho > \frac{1}{2}$ we have downward parabola with maximum at $w = \frac{1}{2}$ and minima on the ends of the interval $[0,1]$, which implies that we have cascade if $\rho > \frac{z_1}{z_2(1-P)}$. Combining both parts we arrive to the following condition: $\frac{z_2}{z_1} > \frac{1}{(1-P)}\min\{2, \rho^{-1}\}$.

Note that the condition on network structure becomes less restrictive when price decreases. Thus if price is a part of decision process the monopolist can achieve highest diffusion when $P = 0$ and condition becomes $\frac{z_2}{z_1} > \min\{2, \rho^{-1}\}$.

Proof of Proposition 2

Substituting constraints to the objective function and deriving with respect to w we find:

$$(1-P)^2\frac{z_1^4(1-2w)(1-2\rho)(2z_1 + z_2(1-P)(1-2\rho))}{2(z_1^2 - z_1z_2\rho(1-P) - wz_2^2(1-P)^2(1-w)(1-2\rho))^2}$$

A denominator of the condition is always positive and thus sign depends on the numerator. Recall that we assume that we are in sub-critical phase with $\frac{z_2}{z_1} < 2(1-P)^{-1}$ and thus term $2z_1 + z_2(1-P)(1-2\rho)$ is always positive. The sign of the condition depends exclusively on values of ρ and w. Namely if $\rho < \frac{1}{2}$ derivative is positive for $w < \frac{1}{2}$ and negative afterwards. Thus, we can conclude that for $\rho < \frac{1}{2}$ objective function has unique maximum at the point $w = \frac{1}{2}$. In the case when $\rho > \frac{1}{2}$ results are reversed and the objective function has its minimum at a point $w = \frac{1}{2}$ and maxima lie on the boundaries, namely $w^* \in \{0, 1\}$. If $\rho = \frac{1}{2}$ all interval $[0, 1]$ satisfies first order condition.

Proof of Proposition 3

We analyze the second part of the functional form of demand. Results for the first part can be obtained by substituting $\rho = \frac{1}{2}$. The demand is decreasing and convex in P:

$$\frac{\partial}{\partial P} Q(P, \rho, z_1, z_2) = -\frac{1}{2}\left(1 + \frac{(1-P)z_1^2\rho(2z_1 - z_2(1-P)\rho)}{(z_1 - (1-P)z_2\rho)^2}\right) < 0$$

The second derivative:

$$\frac{\partial^2}{\partial P^2} Q(P, \rho, z_1, z_2) = \frac{z_1^4\rho}{(z_1 - (1-P)z_2\rho)^3} > 0$$

It is positive since by condition of no global cascade from Lemma 2 we know that $z_1 > (1-P)z_2\rho$. Moreover cross derivative of $Q(P, \rho, z_1, z_2)$ with respect to P and ρ is $-\frac{(1-P)z_1^4}{(z_1-(1-P)z_2\rho)^3}$, which is negative.

Lets turn to the elasticity of demand:

$$\begin{aligned} E_d &= \frac{\partial_P \log Q(P, \rho, z_1, z_2)}{\partial_P \log P} \\ &= -\frac{P}{1-P}\left(1 + z_1\left(\frac{1}{z_1 - (1-P)z_2\rho} - \frac{1}{z_1 + (1-P)(z1^2 - z2)\rho}\right)\right) \end{aligned}$$

Taking derivative of $|E_d|$ with respect to ρ we obtain:

$$\frac{\partial}{\partial \rho}|E_d| = \frac{z_1^3 z_2 P(1-P)^2(z1^2 - z2)\rho^2 + z_1^5 P}{(z1 - (1-P)z_2\rho)^2(z_1 - (1-P)z_2\rho + z_1^2\rho(1-P))^2} > 0$$

Which implies that elasticity of demand is increasing in ρ.

$$\frac{\partial}{\partial \rho} s^*(P, \rho, z_1, z_2) = \frac{(1-P)^2 z_1^3}{2(z_1 - (1-P)z_2\rho)^2} > 0$$

Thus for $\rho > \frac{1}{2}$ function is increasing in ρ.

Proof of Proposition 4

Price is decreasing in the homophily level

The first order condition for P is:

$$\frac{(1-2P)z_1^2-(1-P)z_1[(1-P)z_2+(1-3P)(z_2-z_1^2)]\rho}{2(z_1-(1-P)z_2\rho)^2}-\\ -\frac{(1-P)^2(1-2P)(z_1^2-z_2)z_2\rho^2}{2(z_1-(1-P)z_2\rho)^2}=0$$

Let us fix expected number of friends z_1 and z_2 and call the expression on the left hand side $F(P,\rho)$. The second derivative of $F(P,\rho)$ with respect to P is:

$$F''_P(P,\rho)=\frac{3z_1^4\rho(z_1-z_2\rho)}{(z_1-(1-P)z_2\rho)^4}>0$$

It is positive since by conditions of no giant component we have $z_1>z_2\rho$. Thus function is convex. Evaluating function on the ends of the interval we have $F(0,\rho)=\frac{z_1+z_1^2\rho-z_2\rho}{2(z_1-z_2\rho)}>0$ and $F(1,\rho)=-\frac{1}{2}$. The first derivative with respect to P is negative at 0:

$$F'_P(0,\rho)=-1-\frac{z_1^2\rho(2z_1-z_2\rho)}{(z_1-z_2\rho)^2}<0$$

If $F(P,\rho)$ is convex in P, positive at 0 and negative at 1, we can conclude that function should intersect x-axis from above on the interval $[0,1]$. Hence, the derivative of the $F(P,\rho)$ evaluated for the optimal price $P=P^*$ is negative, $\frac{\partial}{\partial P}F(P^*,\rho)<0$.

Moreover $F(\frac{1}{2},\rho)=\frac{z_1^3\rho}{2(2z_1-z_2\rho)^2}>0$, which implies that the optimal price is always less than $\frac{1}{2}$. The derivative with respect to ρ is:

$$F'_\rho(P,\rho)=-\frac{(1-P)z_1^3\left[(1-P)^2z_2\rho-(1-3P)z_1\right]}{2(z_1-(1-P)z_2\rho)^3}$$

The sign of the derivative depends on the sign of the term in square brackets. Thus taking into account that $P>0$ the derivative is negative if $P>\bar{P}=1-\frac{3z_1}{2z_2\rho}+\frac{\sqrt{9z_1^2-8z_1z_2\rho}}{2z_2\rho}$. One can check that $F(\bar{P},\rho)>0$. The fact that $F(P,\rho)$ intersects x-axes from above, implies that $P^*>\bar{P}$ and thus the derivative is negative.

By implicit function theorem we know:

$$\frac{\partial P^*}{\partial\rho}=-\left.\frac{\frac{\partial}{\partial\rho}F(P,\rho)}{\frac{\partial}{\partial P}F(P,\rho)}\right|_{P=P^*}$$

Taking into account that $F'_P(P^*,\rho)<0$ and $F'_\rho(P^*,\rho)<0$ we can conclude that $\frac{\partial P^*}{\partial\rho}$ is negative and consequently the optimal price P^* is decreasing in ρ.

Price is decreasing in z_2

Similar to previous analysis we hold fix z_1 and z_2 and consider function $F(P,\rho)$. The derivative of $F(P,\rho)$ with respect to z_2 is:

$$F'_{z_2}(P,\rho) = (1-P)^2 z_1^2 \rho^2 \frac{(1-4P)z_1 - (1-P)(1-2P)z_2\rho}{2(z_1-(1-P)z_2\rho)^3}$$

Taking into account that $P > 0$, the expression is negative if and only if $P > \bar{P} = \frac{-4z_1+3z_2\rho+\sqrt{16z_1(z_1-z_2\rho)+z_2^2\rho^2}}{4z_2\rho}$. One can check that $F(\bar{P},\rho) > 0$. The fact that $F(P,\rho)$ intersects x-axes from above, implies that $P^* > \bar{P}$ and thus $F'_{z_2}(P^*,\rho) < 0$. Using the implicit function theorem and the fact that $F'_P(P,\rho) < 0$ we can conclude that $\frac{\partial P^*}{\partial z_2} < 0$

Profits are increasing in the level of homophily

Lets take two levels of homophily $\rho_2 > \rho_1$. By the Proposition 3 we know that for any fixed price P following holds $Q(P,\rho_2,z_1,z_2) > Q(P,\rho_1,z_1,z_2)$. Thus for any given price P the same is true for profits, namely $PQ(P,\rho_2,z_1,z_2) > PQ(P,\rho_1,z_1,z_2)$. Assume further that P_1^* is optimal price for ρ_1. The previous result states that $\pi(\rho_2,P_1^*) > \pi(\rho_1,P_1^*)$ and thus by optimality we know that $\pi(\rho_2,P_2^*) > \pi(\rho_2,P_1^*) > \pi(\rho_1,P_1^*)$, where P_2^* is optimal price for ρ_2.

Proof of Proposition 7

A derivative of sales function with respect to product characteristic is given by:

$$s'(w) = (1-P)\times$$

$$\times\frac{1+z_1(1-P)(1-3\rho-(1-2\rho)[w^2(1-P)z_1+2w(1-\rho(1-P)z_1)+\rho(1-P)z_1])}{(1-wz_1^2(1-P)^2(1-2\rho)(1-w)-z_1\rho(1-P))^2}$$

Note that the denominator is always positive. It is easy to verify that for $\rho > \frac{1}{2}$ all terms in the numerator involving w are positive too. Thus if we prove that $s'(0) > 0$ then the derivative of sales function with respect to w is positive on the whole interval $[0,1]$ and we can conclude that the optimal design is $w^* = 1$. Substituting $w = 0$ into the derivative and taking into account that $z_1 < \frac{1}{\rho(1-P)}$ we have:

$$s'(0) = \frac{1+(1-P)z_1(1-2\rho)}{1-(1-P)z_1\rho} = 1 + \frac{(1-P)z_1(1-\rho)}{1-(1-P)z_1\rho} > 0$$

Thus for $\rho > \frac{1}{2}$ characteristic $w^* = 1$ is the solution to the problem.

When $\rho < \frac{1}{2}$ then all terms involving w are negative and numerator is decreasing function in w. Thus numerator has its minimum at $w = 1$ and condition for $s'(w) > 0$ on the interval $w \in [0,1]$ is simply $s'(1) > 0$. This in turn implies that if $w^* = 1$ is maximum it is also the global maximum. The derivative at 1 is greater than zero if:

$$-2(1-P)^2 z_1^2 \rho^2 + z_1(1-P+3(1-P)^2 z_1)\rho + 1 - (1-P)z_1 - (1-P)^2 z_1^2 > 0$$

An expression on the left describes downward sloping parabola. The solution is:

$$\rho_1 = \frac{3}{4} + \frac{1 - \sqrt{9 - 2z_1(1-P) + (1-P)^2 z1^2}}{4(1-P)z_1}$$

$$\rho_2 = \frac{3}{4} + \frac{1 + \sqrt{9 - 2z_1(1-P) + (1-P)^2 z1^2}}{4(1-P)z_1}$$

It can be shown that $\rho_2 > 1$ and thus condition reduces to:

$$\frac{3}{4} + \frac{1 - \sqrt{9 - 2z_1(1-P) + (1-P)^2 z1^2}}{4(1-P)z_1} < \rho < \frac{1}{2}$$

Combining the previous condition with the case of $\rho > \frac{1}{2}$, we know that $w^* = 1$ is solution if:

$$\rho > \hat{\rho}_T(z_1, P) = \frac{3}{4} + \frac{1 - \sqrt{9 - 2z_1(1-P) + (1-P)^2 z1^2}}{4(1-P)z_1}$$

On the other hand for $\rho < \hat{\rho}_T(z_1, P)$ there is an interior solution which is given by:

$$w^* = \rho - \frac{1 - 2\rho - \sqrt{(1-\rho)(1-2\rho)(1-(1-P)z_1\rho)[2+(1-P)z_1(1-2\rho)]}}{(1-P)z_1(1-2\rho)}$$

It is interesting to note that $w = \frac{1}{2}$ is never solution for $\rho < \frac{1}{2}$ since $s'(\frac{1}{2}) = \frac{4(1-P)}{(2-(1-P)z_1)(2+(1-P)z_1(1-2\rho))} > 0$, which implies that $w^* > \frac{1}{2}$.

Proof of Proposition ??

Lets denote by $\theta = \frac{z_2}{z_1}$ and by $\lambda(w)$ the following polynom:

$$\lambda(w) = 1 - (1-P)^2 w(1-w^r)^{\frac{1}{r}} \theta^2 (1-2\rho) - (1-P)\left(w + (1-w^r)^{\frac{1}{r}}\right)\theta\rho$$

The global cascade of sales occurs if there is $\bar{w}$ such that $\lambda(\bar{w}) \leq 0$. One can readily obtain condition for ρ. We just need to take the derivative with respect to ρ and to show that it is negative. Thus there is the global cascade of sales if $\rho > \hat{\rho}(\theta, r)$, where

$$\hat{\rho}(\theta, r) = \min_{0 \leq w \leq 1} \frac{1 - w\theta^2(1-P)^2(1-w^r)^{\frac{1}{r}}}{(1-P)\theta\left(w + (1-w^r)^{\frac{1}{r}}(1-2w\theta(1-P))\right)}$$

From the previous analysis we know that candidates for maxima are extreme values and such that $q^A = q^B$, which in our case is $\left(\frac{1}{2}\right)^{\frac{1}{r}}$. Evaluating polynomial at 0 we have $\lambda(0) = 1 - (1-P)\theta\rho$ and at the point $\left(\frac{1}{2}\right)^{\frac{1}{r}}$ we have:

$$\lambda\left(2^{-\frac{1}{r}}\right) = 4^{-\frac{1}{r}}(2^{\frac{1}{r}} - (1-P)\theta)\left[2^{\frac{1}{r}} + (1-P)\theta(1-2\rho)\right]$$

From the first condition we can conclude that if $\rho > \frac{1}{\theta}$ then there is global cascade. From the second we see that if $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$ and $\rho < \hat{\rho} = \frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)}$ then global cascade of sales arises. Lets consider a case when $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$, but $\rho < \frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)}$. From the first condition we know that there is global cascade if $\rho > \frac{1}{\theta(1-P)}$ thus to insure existence of the global cascade we should prove that $\frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)} > \frac{1}{\theta(1-P)}$.

$$\frac{1}{2}(1-P) + 2^{\frac{1-r}{r}}\theta^{-1} > \theta^{-1}$$

$$\frac{1}{2}(1-P) > \theta^{-1}\left(1 - 2^{\frac{1-r}{r}}\right)$$

$$\theta > (1-P)^{-1}\left(2 - 2^{\frac{1}{r}}\right)$$

$$\theta > 2^{\frac{1}{r}}(1-P)^{-1}\left(2^{\frac{r-1}{r}} - 1\right)$$

There is the global cascade if the former condition holds. However, we have assumed that $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$, which implies that former condition holds, since $\left(2^{\frac{r-1}{r}} - 1\right) < 1$. Thus we have shown that if $\theta > 2^{\frac{1}{r}}$ there is global cascade independently of the homophily level ρ.

Proof of Proposition 8

For the case of Poisson degree distribution size of sales cascade is given by:

$$s(w, P, r, z_1) = \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

The product characteristic $w^* = 0$ is global maximum if for any w, $s(0) \geq s(w)$:

$$\frac{1-P}{2(1-z_1\rho(1-P))} \geq \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

since $1-P\geq 0$ by definition, we have

$$\frac{1}{2(1-z_1\rho(1-P))}-\frac{(w+(1-w^r)^{\frac{1}{r}}(1+2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho+(1-w^r)^{\frac{1}{r}}(\rho+wz_1(1-P)(1-2\rho))))}\geq 0$$

Note further that denominators of two fractions are positive due to condition of no global cascade, thus the sign of the expression depends on the numerator of combined terms, which is:

$$-4(1-P)^2w(1-w^r)^{\frac{1}{r}}z_1^2\rho^2+4(1-P)w(1-w^r)^{\frac{1}{r}}z_1(1+(1-P)z_1)\rho+$$
$$+1-w-(1-w^r)^{\frac{1}{r}}(1+(1-P)wz_1(2+(1-P)z_1))\quad\geq 0$$

Note that expression describes downward sloping parabola and thus our condition will be in the form $\rho_1\leq\rho\leq\rho_2$, where ρ_1 and ρ_2 are solutions to the quadratic equation:

$$\rho_1=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{z_1(1-P)}-\frac{1}{z_1(1-P)}\sqrt{\frac{(1-w)(1-(1-w^r)^{\frac{1}{r}})}{w(1-w^r)^{\frac{1}{r}}}}\right)$$

$$\rho_2=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{z_1(1-P)}+\frac{1}{z_1(1-P)}\sqrt{\frac{(1-w)(1-(1-w^r)^{\frac{1}{r}})}{w(1-w^r)^{\frac{1}{r}}}}\right)$$

The condition should hold for all w and thus we should find the maximum of ρ_1 and the minimum of ρ_2. In order to do this we should identify maximum and minimum of the term with w. Taking derivative of this term with respect to w we have:

$$-\frac{(1-w^r)^{-\frac{1+r}{r}}\left[(1-2w^r)+\left(w^{r+1}-(1-w^r)^{\frac{1+r}{r}}\right)\right]}{w^2}$$

Independently of r one can see that first derivative is zero at the point $w=\left(\frac{1}{2}\right)^{\frac{1}{r}}$. It can be proved that for $r<1$, $-(1-2w^r)-\left(w^{r+1}-(1-w^r)^{\frac{1+r}{r}}\right)$ is negative for $w<\left(\frac{1}{2}\right)^{\frac{1}{r}}$ and positive afterwards, which implies that minimum is at $w=\left(\frac{1}{2}\right)^{\frac{1}{r}}$ and maxima lay at borders.

Substituting back values of w into condition for ρ we obtain:

$$\frac{1}{2}-\frac{2^{\frac{1}{r}}-2}{2z_1(1-P)}<\rho<\frac{1}{2}+\frac{2^{\frac{1}{r}}}{2z_1(1-P)}$$

Since $z_1\leq 2^{\frac{1}{r}}$ the minimum of last term is 1. Taking it into account we can rewrite the condition as:

$$\frac{1}{2} - \frac{2^{\frac{1}{r}} - 2}{2z_1(1-P)} < \rho < 1$$

On the other hand for $r > 1$, one can show that $w^* = 0$ is never a solution. Lets evaluate derivative of sales at $w = 0$ for the case when $r > 1$ we have:

$$\left.\frac{\partial s}{\partial w}\right|_{w=0} = \frac{(1-P)(1+(1-P)z_1(1-2\rho))^2}{2(1-(1-P)z_1\rho])^2} > 0$$

It is always positive which implies that $w^* = 0$ is never solution for $r > 1$.

The symmetric design $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ is global maximum if for any w, $s\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}\right) \geq s(w)$:

$$\frac{1-P}{2^{\frac{1}{r}} - (1-P)z_1} \geq \frac{(1-P)(w+(1-w^r)^{\frac{1}{r}}(1+2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho+(1-w^r)^{\frac{1}{r}}(\rho+wz_1(1-P)(1-2\rho))))}$$

The denominators are positive due to no global cascade condition and thus sign of the expression depends on the numerator of combined fraction, which is:

$$\begin{aligned} -2(1-P)(w+(1-2^{\frac{1+r}{r}}w)(1-w^r)^{\frac{1}{r}})z_1\rho + 2 - (2-2w^r)^{\frac{1}{r}} + \\ +(1-P)(1-2^{\frac{1+r}{r}}w)(1-w^r)^{\frac{1}{r}}z_1 - w(2^{\frac{1}{r}} - (1-P)z_1) \quad \geq 0 \end{aligned}$$

Note that the line is downwards sloping if for any w, $(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}}) > 0$. Thus to prove that it has downward slope we should prove that the minimum of the term $w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}}$ is greater or equal to 0. The first derivative is:

$$1 - 2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} - \left(\frac{(1-w^r)^{\frac{1}{r}}}{w}\right)^{1-r} \tag{4}$$

It is zero at point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. For $r < 1$ the expression is negative for $w < \left(\frac{1}{2}\right)^{\frac{1}{r}}$, since term $2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} > 0$ and term $w^{-(1-r)}(1-w^r)^{\frac{1-r}{r}} > 1$ (by properties of frontier). The expression is positive for $w > \left(\frac{1}{2}\right)^{\frac{1}{r}}$, because term $2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} < 0$ and term $w^{-(1-r)}(1-w^r)^{\frac{1-r}{r}} < 1$. This implies that minimum lies at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ where the expression equals to zero. Thus the line has negative slope. And condition becomes:

$$\rho < \hat{\rho}_1 = \min_w \left\{ \frac{1}{2}\left(1 - \frac{2^{\frac{1}{r}}(w+(1-w^r)^{\frac{1}{r}}) - 2}{z_1(1-P)(w+(1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}})}\right)\right\}$$

we can show that for $r < 1$ the expression with w has its maxima on the borders and thus, evaluating at $w = 0$ we have:

$$\rho < \frac{1}{2} - \frac{2^{\frac{1}{r}} - 2}{2z_1(1-P)}$$

The case when $r > 1$

Lets rewrite the expression (4):

$$1 - (2^{\frac{1+r}{r}}(1-2w^r) + w^{r-1})(1-w^r)^{\frac{1-r}{r}}$$

It is zero at point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. For $r > 1$ and $w \leq \left(\frac{1}{2}\right)^{\frac{1}{r}}$, $\min\{(1-w^r)^{\frac{1-r}{r}}\} = 2^{\frac{r-1}{r}}$ and $\min\{(2^{\frac{1+r}{r}}(1-2w^r) + w^{r-1})\} = 2^{\frac{1-r}{r}}$. Thus minimum of the product of two terms is equal to 1 and this implies that for $w \leq \left(\frac{1}{2}\right)^{\frac{1}{r}}$ expression is negative. For $w > \left(\frac{1}{2}\right)^{\frac{1}{r}}$, $\max\{(1-w^r)^{\frac{1-r}{r}}\} = 2^{\frac{r-1}{r}}$ and $\max\{(2^{\frac{1+r}{r}}(1-2w^r) + w^{r-1})\} = 2^{\frac{1-r}{r}}$, which implies that expression is positive. Thus minimum of the expression is at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ line has negative slope.

This leads us again to the condition:

$$\rho < \hat{\rho}_2 = \min_w \left\{ \frac{1}{2} \left(1 - \frac{2^{\frac{1}{r}}(w + (1-w^r)^{\frac{1}{r}}) - 2}{z_1(1-P)(w + (1 - 2^{\frac{r+1}{r}} w)(1-w^r)^{\frac{1}{r}})} \right) \right\}$$

Thus we can establish that there is $\hat{\rho}_2$ such that if $\rho < \hat{\rho}_2$ than the optimal characteristic is $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. Note that condition $\hat{\rho}_1 \neq \hat{\rho}_2$ since we optimize for different values of r. If $\rho > \hat{\rho}_2$ then solution belongs to $(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1)$, since as we have seen $w = 1$ is always not optimal for the case of $r > 1$.

Proof of Lemma 3

Let us denote by $x = \rho u^A + (1-\rho)u^B$ and by $y = \rho u^B + (1-\rho)u^A$ than we can rewrite system of equations as following:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$u^A = 1 - q^A + q^A \hat{G}_1[x]$$
$$u^B = 1 - q^B + q^B \hat{G}_1[y]$$

Or equivalently:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$x = \rho[(1-q^A) + q^A\hat{G}_1(x)] + (1-\rho)[(1-q^B) + q^B\hat{G}_1(y)]$$
$$y = \rho[(1-q^B) + q^B\hat{G}_1(y)] + (1-\rho)[(1-q^A) + q^A\hat{G}_1(x)]$$

Substituting $\gamma = \frac{1}{2}$ we obtain following maximization problem of the monopolist:

$$\max_{q^A, q^B} \frac{1}{2}\left[q^A + q^B - q^A G_0(x) - q^B G_0(y)\right]$$

s.t.

$$x = 1 - \rho q^A - (1-\rho)q^B + \rho q^A \hat{G}_1(x) + (1-\rho)q^B\hat{G}_1(y)$$
$$y = 1 - (1-\rho)q^A - \rho q^B + (1-\rho)q^A\hat{G}_1(x) + \rho q^B \hat{G}_1(y)$$

Proof of Proposition 9

The FOC of the monopolist problem is:

$$-G_0(x) - wG_0'(x)\frac{\partial x}{\partial w} + G_0(y) - (1-w)G_0'(y)\frac{\partial y}{\partial w} = 0$$

The derivatives of constraints with respect to w are following:

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

The case when firm's action has no effect.

Interesting case arises when $\rho = \frac{1}{2}$. It seems that w has no effect on the size of global cascade of sales . Substituting $\rho = \frac{1}{2}$ we can rewrite the problem as following:

$$\max_w 1 - wG_0(x) - (1-w)G_0(y)$$

s.t.

$$x = \tfrac{1}{2} + \tfrac{1}{2}w\hat{G}_1(x) + \tfrac{1}{2}(1-w)\hat{G}_1(y)$$
$$y = \tfrac{1}{2} + \tfrac{1}{2}w\hat{G}_1(x) + \tfrac{1}{2}(1-w)\hat{G}_1(y)$$
$$0 \le w \le 1, 0 \le x \le 1, 0 \le y \le 1$$

Note that in this case $x = y$ for any w. This implies that maximization problem of the monopolist in the case of $\rho = \frac{1}{2}$ does not depend on w:

$$\max_{w} 1 - G_0(x)$$

s.t.

$$x = \tfrac{1}{2} + \tfrac{1}{2}\hat{G}_1(x)$$
$$0 \le x \le 1$$

Thus eventual outbreak is the same for all values of w and moreover it's size is equal to the giant component of connected consumers.

The case when specialized design is optimal.

We want to check when it is optimal to focus on the first group or equivalently when $w = 1$ is the solution. Note that $w = 1$ is corner solution that is why it is enough to show that derivative of $\frac{\partial s}{\partial w}\big|_{w=1}$ is non-negative:

$$-G_0(x) - G_0'(x)\frac{\partial x}{\partial w} + G_0(y) > 0$$

s.t.

$$x = 1 - \rho + \rho\hat{G}_1(x)$$
$$y = \rho + (1-\rho)\hat{G}_1(x)$$

The derivative of first constraint with respect to w is:

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y)$$

Thus we have

$$\frac{\partial x}{\partial w} = \frac{1 - 2\rho + \rho\hat{G}_1(x) - (1-\rho)\hat{G}_1(y)}{1 - \rho\hat{G}_1'(x)}$$

Substituting it to the maximization problem we obtain:

$$G_0(y) - G_0(x) - G_0'(x)\frac{1 - 2\rho + \rho\hat{G}_1(x) - (1-\rho)\hat{G}_1(y)}{1 - \rho\hat{G}_1'(x)} > 0$$

s.t.

$$x = 1 - \rho + \rho\hat{G}_1(x)$$
$$y = \rho + (1-\rho)\hat{G}_1(x)$$

Let us rewrite the first equation as:

$$[G_0(y) - G_0(x)] + G_0'(x)\frac{\rho(1 - \hat{G}_1(x)) - (1-\rho)[1 - \hat{G}_1(y)]}{1 - \rho\hat{G}_1'(x)} \ge 0$$

The first term is non-negative when $y \geq x$ and the condition is following:

$$\rho + (1-\rho)\hat{G}_1(x) \geq 1 - \rho + \rho\hat{G}_1(x)$$

$$(2\rho - 1)[1 - \hat{G}_1(x)] \geq 0$$

since $\hat{G}_1(x) \leq 1$ for all $x \in [0,1]$ the condition is $\rho \geq \frac{1}{2}$.

The same happens with the second term when $\rho > \frac{1}{2}$. Note that $\rho > \frac{1}{2}$ implies that $\hat{G}_1[x] < \hat{G}_1[y]$ consequently $1-\hat{G}_1[x] > 1-\hat{G}_1[y]$. Multiplying both sides by ρ and taking into account that $\rho \geq 1-\rho$ for $\rho \geq \frac{1}{2}$ we have:

$$\rho[1-\hat{G}_1(x)] \geq \rho[1-\hat{G}_1(y)] \geq (1-\rho)[1-\hat{G}_1(y)]$$

Thus we have proved that $w = 1$ is locally optimal if $\rho > \frac{1}{2}$ independently of degree distribution.

The case when symmetric design is optimal

The symmetric design $w = \frac{1}{2}$ is optimal if following holds:

$$-G_0(x) - \frac{1}{2}G_0'(x)\frac{\partial x}{\partial w} + G_0(y) - \frac{1}{2}G_0'(y)\frac{\partial y}{\partial w} = 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\rho\hat{G}_1(x) + \frac{1}{2}(1-\rho)\hat{G}_1(y)$$

$$y = \frac{1}{2} + \frac{1}{2}(1-\rho)\hat{G}_1(x) + \frac{1}{2}\rho\hat{G}_1(y)$$

and

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

It is easy to check that $x = y$ satisfies our conditions on x and y, thus we have:

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

Moreover it is possible to show (it should be) that system of equations for x and y has only two solution. The first one is $x = y = 1$. Thus we have:

$$G_0'(x)\left[\frac{\partial x}{\partial w}+\frac{\partial y}{\partial w}\right]=0$$

s.t.

$$x=\frac{1}{2}+\frac{1}{2}\hat{G}_1(x)$$

The first solution to the FOC equation is $G_0'(x) = 0$ which implies $x = 0$. This obviously does not satisfy second equation, thus the only possibility left is $\frac{\partial x}{\partial w} = 0$

$$\frac{\partial x}{\partial w}=1-2\rho+\rho\hat{G}_1(x)+\rho w\hat{G}_1'(x)\frac{\partial x}{\partial w}-(1-\rho)\hat{G}_1(y)+(1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w}=-1+2\rho+(1-\rho)\hat{G}_1(x)+(1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w}-\rho\hat{G}_1(y)+\rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

Solving previous system and substituting $y = x$ we have:

$$\frac{\partial x}{\partial w}=-\frac{1-2\rho-\hat{G}_1(x)+2\rho\hat{G}_1(x)-\rho\hat{G}_1(x)\hat{G}_1'(x)+\rho\hat{G}_1'(x)+\frac{1}{2}\hat{G}_1(x)\hat{G}_1'(x)-\frac{\hat{G}_1'(x)}{2}}{-\frac{1}{2}\rho[\hat{G}_1'(x)]^2+\frac{1}{4}[\hat{G}_1'(x)]^2+\rho\hat{G}_1'(x)-1}$$

$$\frac{\partial y}{\partial w}=\frac{1-2\rho-\hat{G}_1(x)+2\rho\hat{G}_1(x)-\rho\hat{G}_1(x)\hat{G}_1'(x)+\rho\hat{G}_1'(x)+\frac{1}{2}\hat{G}_1(x)\hat{G}_1'(x)-\frac{\hat{G}_1'(x)}{2}}{-\frac{1}{2}\rho[\hat{G}_1'(x)]^2+\frac{1}{4}[\hat{G}_1'(x)]^2+\rho\hat{G}_1'(x)-1}$$

Note that $\frac{\partial x}{\partial w} = -\frac{\partial y}{\partial w}$ and thus we have that:

$$G_0'(x)\left[\frac{\partial x}{\partial w}+\frac{\partial y}{\partial w}\right]=0$$

This implies that $w = \frac{1}{2}$ is always the critical point. What is left to proof is that it is maximum when $\rho < \frac{1}{2}$.

SOC of the problem

$$-2G_0'(x)\frac{\partial x}{\partial w}-wG_0''(x)\left(\frac{\partial x}{\partial w}\right)^2-wG_0'(x)\frac{\partial^2 x}{\partial w^2}+2G_0'(y)\frac{\partial y}{\partial w}-(1-w)G_0''(y)\left(\frac{\partial y}{\partial w}\right)^2-(1-w)G_0'(y)\frac{\partial^2 y}{\partial w^2}$$

SOC when $w = \frac{1}{2}$

$$-\frac{4z(1-2\rho)(1-\hat{G}_1(x))}{\left(2-\hat{G}_1'(x)\right)\left(2+(1-2\rho)\hat{G}_1'(x)-2\right)^2}\times$$

$$\times \left((1-2\rho) \left(\hat{G}_1'(x)^2 + 2\hat{G}_1'(x) + \hat{G}_1''(x)(1-\hat{G}_1(x)) \right) \hat{G}_1(x) + 8\hat{G}_1(x) + (1-2\rho) \left(2 - \hat{G}_1'(x) \right) \hat{G}_1'(x) \right)$$

Thus we can conclude that $w = \frac{1}{2}$ is local maximum for $\rho < \frac{1}{2}$ if:

$$2 - \hat{G}_1'(x) > 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

Let us denote by $F(x) = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$ then solution to equation x^* is such that $F(x)$ crosses 45 degree line in x^* from above since $F(0) = \frac{1}{2}$ Thus we can conclude that $F'(x^*) < 1$. Thus $\frac{1}{2}\hat{G}_1'(x) < 1$ and consequently $\hat{G}_1'(x) < 2$, which in turn implies that our condition always holds.

When $w = \frac{1}{2}$ should be preferred to $w = 1$?

Recall that in the case when $w = \frac{1}{2}$ the size of the giant component is given by:

$$S(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}G_0(x_m^*)$$

$$x_m^* = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x_m^*)$$

On the other hand if $w = 1$ we have:

$$S(1) = \frac{1}{2} - \frac{1}{2}G_0(x_b^*)$$

$$x_b^* = 1 - \rho + \rho\hat{G}_1(x_b^*)$$

Due to the monotonicity of the $G_0(x)$ we know that $S(\frac{1}{2}) > S(1)$ whenever $x_m^* < x_b^*$. So basically we should see how ρ affects solution to fixed point problem, since first equation is just particular case of the second. Using IFT we have:

$$\frac{\partial x}{\partial \rho} = -1 + \hat{G}_1(x) + \rho\hat{G}_1'(x)\frac{\partial x}{\partial \rho}$$

$$\frac{\partial x}{\partial \rho} = -\frac{1 - \hat{G}_1(x)}{1 - \rho\hat{G}_1'(x)}$$

Note that x is solution to fixed point of $1 - \rho + \rho\hat{G}_1(x)$ at x^* it should cross the 45 degree line and this in turn implies that $\rho\hat{G}_1'(x) < 1$ thus we have shown that $\frac{\partial x}{\partial \rho} < 0$. Thus in turn implies that if $\rho < \frac{1}{2}$ $x_b^* > x_m^*$ and thus $S(\frac{1}{2}) > S(1)$ on the other hand if $\rho > \frac{1}{2}$ thus $S(\frac{1}{2}) < S(1)$