"Pricing and Advertising in a Market with Consumer Referrals"

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March 24, 2011

Abstract

In many industries, especially in markets for services, firms reward their customers for making referrals. We analyze the implications of referral policies in a model where (i) a monopoly chooses its price, advertising intensity and a referral fee, and then (ii) consumers decide whether to buy the product and the buyers choose to what extent to refer other consumers to the firm. We provide conditions for monopoly to support active referrals and characterize the equilibrium. Not surprisingly, monopoly advertises less under referrals. Perhaps surprisingly, we find that monopoly does not change its price from the monopoly level in an attempt to manage consumer referrals but instead uses a referral fee. We extend the analysis to the case where consumers know more than monopoly about other consumers' preferences, and therefore referrals are more targeted in equilibrium than advertisements.

Keywords: consumer referral policy, referral bonus, kickbacks, targeted advertising, word of mouth, referral reward program.
JEL numbers: C7, D4, D8, L1.

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1 Introduction

Firms often pay existing customers for referring potential customers to the products or services offered by the firms. Such referral programs are advertised as "Win/Win/Win" because the existing customers, potential customers, and the firm can all benefit from referrals. The recent explosion in consumer-generated media together with the documented trust consumers have in recommendations of other people, explains why firms would strive to manage word of mouth.[1] Indeed, as consumer forums, blogs and other means of consumer interaction are enhanced by advances in technology, the mix of information channels firms rely on is changing. Firms shift away from mass advertising to more targeted channels which rely on consumers to spread the word about their goods. Improved online referral systems promise to make it easier for firms to monitor and control the referral activity. How can firms use these new capabilities in designing their promotional strategies?

Adopting a referral policy is one way firms can try to harness the power of word of mouth (WOM). A consumer referral policy is a promise by a firm to pay its customer a flat fee or a commission for referring other people who become the firm's customers. For example, *DirectTV*'s "Spread the Word Program" offers a $100 credit to any customers for referring a friend who would sign up for a *DirectTV* service. Referral policies are adopted in a variety of industries, including banking, web design services, home remodeling, housing, vacation packages, home alarm systems, and high-speed internet connection. They are used in recruitment of nurses, technicians, and US army personnel, as well as in selling cars and houses. Private schools, doctors, and daycare centers give out referral bonuses as well.

A casual observation of referral policies suggests that firms usually pay re-

[1] The Nielsen Global Online Consumer Survey of April 2009, which surveyed over 25,000 Internet consumers from 50 countries, reports that the most trustworthy information channel are recommendations from personal acquaintances: 90 percent of Internet consumers worldwide trust ("completely" or "somewhat") recommendations from people they know, while 70 percent trust consumer opinions posted online. The only media that experienced a drop in terms of consumer trust is newspaper ads.

ferral bonuses only to their existing customers. That is, consumers have to buy a firm's product in order to be able to collect a referral payment for referring someone else. The referral payments take a form of a fixed fee, a commission, or an in-kind reward.[2] The referral payment is typically made only if the referral customer buys the good. In our analysis, we will focus on the common case of fixed referral fees paid out to existing customers for referring a new customer who buys the product.[3]

We analyze the optimal decisions of monopoly regarding pricing, advertising, and referral activity. We answer such questions as: When would a monopoly support active consumer referrals? Would monopoly set a higher or lower price under referrals? Would monopoly engage in more or less advertising under referrals? What are the overall welfare effects of referral policies?

To answer these questions, we introduce consumer referral policies into a monopoly market with random advertising (e.g., Butters, 1977). Consumers differ in their willingness to pay for the product. They can become informed about the firm's product either directly through random ads or indirectly through consumer referrals. The monopoly firm can manage its referral system in our model. A higher referral fee encourages more referrals and hence more sales, but as referrals expand they become less effective due to referral congestion (as more referrals are given, an additional referral attempt is not likely to pay off). We show that monopoly chooses to rely on an active referral policy as long as the cost to consumers of making referrals is not too high. We find that monopoly advertises less under referrals and it uses referral fee to manage referral activity, keeping the price at the monopoly level. Importantly, whenever a monopoly chooses to support active referrals, this is a Pareto improvement.

In the next section, we analyze consumers' purchase and referral decisions

[2] Referral payments are made with cash, deposit, gift certificate, bonus points, free product or service, or entry into a lottery.

[3] In contrast, in affiliate marketing, referral payments - typically commissions - are given to an affiliate, not a customer. Referral payments are either based on consumer traffic (pay-per-click) or the number of sales consummated. The pay-per-click payment tends to create an incentive problem since referrals may be given to people who do not have an intention to buy the product, and it is more susceptible to fraudulent behavior by competitors.

for a given monopoly policy regarding price, advertising, and consumer referrals. Section 3 describes how monopoly chooses its profit-maximizing policy, anticipating the following equilibrium consumer behavior. Section 4 discusses the social welfare implications of consumer referrals. Section 5 presents alternative formulations of the model. We conclude in Section 6. Appendix A contains the proofs.

2 The Basic Model of Consumer Referrals

In this section, we describe the equilibrium consumer referral behavior in a market served by a monopoly that sets a price, advertises its product, and pays referral bonuses to the existing customers for referring new customers.

Consider a monopoly selling a product to a large number N of consumers at zero unit cost of production. Monopoly policy is described by a triple (p, r, π): monopoly chooses a level of advertisement (a fraction $\pi \in [0,1]$ of consumers reached by advertisements), its price $p \geq 0$, and a referral policy characterized by a referral fee $r \geq 0$. We assume that the monopoly cannot price-discriminate between consumers who receive an advertisement and those who are referred to the firm. Consumers differ only in their willingness to pay, which is assumed to follow a cumulative density function G on $[0,1]$ with a continuous density function g; $G(0) = 0$ and $G(1) = 1$. We assume that $-2g(p) - pg'(p) < 0$ for all $p \in [0,1]$, which guarantees the concavity of the profit function under no referrals. Each consumer purchases at most one unit of the product.

Advertisements are distributed to consumers uniformly at random. That is, the probability that each consumer becomes informed through advertising is independent of the consumer's valuation for the product. The cost of advertising is described by function $C(\pi)$, which is increasing at an increasing rate in the fraction of consumers reached, π, $C'(\pi) > 0$ and $C''(\pi) \geq 0$. Consumers who become informed through ads buy the product at the stated price or remain inactive. Without receiving an advertisement or referral, a consumer would not know about the product and cannot purchase it.

After making a purchase, a consumer can attempt to collect referral fees by referring other people. The decision to make referral attempts is endogenous in the model. A consumer trades off the cost and the expected benefit of making referral attempts. Each referral attempt costs $c > 0$, which captures the cost of contacting and informing a contact about the product. On the benefit side, referral attempts may be successful or unsuccessful. We initially assume that a referrer does not know other people's willingness-to-pay and whether they have been informed through advertisements. If a referrer's contact has a low willingness-to-pay and/or are already informed, the referral attempt will not be successful. Furthermore, potential referrals may have received referral attempts from others and may choose a different referring person. Referring consumers simultaneously and independently choose referral intensity, i.e. the fraction of consumers to refer. They send referrals at random but without contacting the same person more than once, and referrals sent by different referrers are independently distributed among all consumers. A consumer choosing an individual referral intensity q contacts qN distinct consumers, but some of these consumers may have received advertisements or may have received multiple referral attempts.

Since monopoly does not price discriminate based on a consumer's information source, waiting to buy by referral can never be better for a consumer than buying immediately. Therefore, it is assumed that if a consumer receives an advertisement and has a nonnegative benefit from the product, she purchases it rather than waits for a referral to purchase by referral. The number of consumers who receive and do not receive ads is πN and $(1-\pi)N$, respectively. The number of informed consumers who purchase the product and are potential referrers is $n = (1-G(p))\pi N$. For any referrals to be given in the market, we need to assume that the number of referrers and well as the number of uninformed consumers are positive, that is, $n = (1-G(p))\pi N > 0$ and $m = (1-\pi)N > 0$. This implies that $\pi \in (0, 1)$ and $1 - G(p) > 0$ must hold for any referral activity to be supported in the market, and we assume these constraints hold when we characterize the equilibrium.

A *consumer referral equilibrium* is a strategy profile $\mathbf{q}^* = (q_1^*, ..., q_n^*) \in [0,1]^n$ such that q_i^* is the best response to $\mathbf{q}_{-i}^*$ for all $i = 1, ..., n$. The equilibrium referral intensity is described by $S^* \equiv \sum_{i=1}^{n} q_i^*$. In the *symmetric consumer referral equilibrium*, $q_i^* = q^*$ for all $i = 1, ..., n$, i.e. each of n referrers sends referrals to a fraction q^* of the total consumer population. Denote by $S = q^* n \geq 0$ the (overall) referral intensity, which is the total number of referral messages send by referrers as a fraction of N.

Referral reach is a fraction of consumers reached by referrals. It is described by $R = 1 - (1-q)^n$. For large n, $R = R(S) \simeq 1 - e^{-S}$ is a function of the (overall) referral intensity $S = qn$. We will assume that n is large and therefore we will use

$$R(S) = 1 - e^{-S} \tag{1}$$

with $R(S) \in [0, 1)$, $R(0) = 0$, $R' > 0$, $R'' < 0$, and $\lim\limits_{S \to \infty} R(S) = 1$. As referral intensity increases, an increasingly smaller fraction of referrals are successful. The *level of congestion* in referral messages can be measured by a function

$$\varphi(S) \equiv \frac{S}{R(S)} = \frac{S}{1 - e^{-S}}, \tag{2}$$

which is a ratio of the number of referral messages sent by referrers to the number of referrals registered by consumers. Note that $\varphi(S) \geq 1$, $\varphi(0) = 1$, $\varphi' > 0$ and $\varphi'' > 0$. Alternatively, we can define a *pass-through rate* as $\frac{R(S)}{S}$.

Proposition 1 demonstrates that there exists a unique symmetric consumer referral equilibrium and characterizes the equilibrium referral intensity.[4]

Proposition 1. *Suppose monopoly chooses a triple* (p, r, π)*. Then there is a unique symmetric consumer referral equilibrium characterized by the referral intensity* $S^* = S^*(p, r, \pi)$ *such that (i) for all* $r \leq r_0 \equiv \frac{c}{(1-\pi)(1-G(p))}$*,* $S^* = 0$ *holds, and (ii) for all* $r > r_0$*,* $S^* = S^*(p, r, \pi)$ *is implicitly defined by*

$$(1 - \pi)(1 - G(p))r = c\varphi(S^*). \tag{3}$$

[4] There is also a continuum of asymmetric equilibria, but the equilibrium referral intensity is identical in all equilibria.

The equilibrium referral intensity $S^* = S^*(p, r, \pi)$ *satisfies* $\frac{\partial S^*}{\partial \pi} < 0$, $\frac{\partial S^*}{\partial p} < 0$, $\frac{\partial S^*}{\partial r} > 0$, *and* $\frac{\partial S^*}{\partial c} < 0$ *for* $r > r_0$.

In the equilibrium, referrers must be indifferent between sending and not sending another referral. That is, each referrer is indifferent among $q_i \in [0, 1]$, and the benefit each referring consumer obtains from making referrals equals its cost. The benefit of giving an additional referral is the referral fee multiplied by the probability that the referral is registered by an uninformed consumer willing to buy the product, $b = (1 - \pi)(1 - G(p))r$, equals the cost of reaching a consumer with a referral message, $c\varphi(S^*)$, which depends on the level of referral congestion. The same property holds at the aggregate level. The total cost to consumers from giving referrals equals the referral fees they collect, $NS^*c = N(1 - \pi)(1 - G(p))R^*r$. The net benefit to consumers from making referrals is zero. The equilibrium referral reach is $R^* = R^*(p, r, \pi) = R^*(S^*(p, r, \pi)) = 1 - e^{-S^*(p,r,\pi)}$. The comparative statics results of Proposition 1 imply that the equilibrium referral intensity S^*, reach R^*, and congestion $\varphi(S^*) = \frac{S^*}{R^*}$ are higher when referral cost, price, and advertising intensity are lower and referral fee is higher. Note that as long as n is large, S^*, R^*, and $\varphi(S^*)$ are independent of n.

Lemma 1 will be useful in the next section, where we analyze the optimal advertising, referral, and pricing policies of the monopoly.

Lemma 1. *Suppose monopoly chooses a triple* (p, r, π). *Then, in the consumer referral equilibrium, (i)* $\frac{dR^*}{dp} / \frac{dR^*}{dr} = -hr$, *where* $h = h(p) = \frac{g(p)}{1-G(p)}$ *is the hazard function, and (ii)* $\frac{dR^*}{d\pi} / \frac{dR^*}{dr} = -\frac{r}{1-\pi}$.

Lemma 1 describes the marginal rates of substitution between the three instruments that the monopoly can use to manage consumer referrals. It implies that in the consumer referral equilibrium, the effective marginal benefit of each variable on the referral reach is the same: $-\frac{dR^*}{d\pi}(1 - \pi) = -\frac{dR^*}{dp}\frac{1}{h} = r\frac{dR^*}{dr}$.

3 Monopoly Choice of Price, Advertising, and Referral Fee

In this section, we characterize the optimal (profit-maximizing) monopoly policy (p^*, r^*, π^*) and conditions under which monopoly would choose to support consumer referrals. In the next section, we will compare the optimal monopoly policy to the socially optimal policy (p^W, r^W, π^W).

Let us first derive the monopoly profits. The number of uninformed consumers who receive referrals is $(1-\pi)R^*N$, where $R^* = R^*(p, r, \pi) = R^*(S^*(p, r, \pi)) = 1 - e^{-S^*(p,r,\pi)}$ is the referral reach in the referral equilibrium characterized by referral intensity $S^*(p, r, \pi)$. Then, the demand the firm faces is

$$\begin{aligned} D(p) &= n(p, \pi) \\ &\quad + m(\pi)R^*(p, r, \pi)(1 - G(p)). \end{aligned} \tag{4}$$

and the monopolist's profit function is

$$\Pi(p, r, \pi) = pn(p, \pi) + (p - r)(1 - G(p))m(\pi)R^*(p, r, \pi) - C(\pi), \tag{5}$$

where $R^* = R^*(p, r, \pi) > 0$ for $r > r_0 \equiv \frac{c}{(1-\pi)(1-G(p))}$ and $R^* = 0$ otherwise.

Recall that the number of uninformed consumers is $m = m(\pi) = (1 - \pi)N$ and the number of informed referring consumers is $n = n(p, \pi) = (1 - G(p))\pi N$. Monopoly profit with an active referral policy can be written as

$$\Pi(p, r, \pi) = N[p(1 - G(p))\pi + (p - r)(1 - G(p))(1 - \pi)R^*(\pi, r, p)] - C(\pi), \tag{6}$$

and it is $\Pi(p, r, \pi) = Np(1 - G(p))\pi - C(\pi)$ when there are no referrals. Clearly, the monopoly price, $p^m \equiv \arg\max_p (p(1 - G(p))$, maximizes the first term in the square brackets. However, it is not clear if the optimal price p^* under consumer referrals is higher or lower than p^m because $\frac{\partial R}{\partial S} > 0$ and $\frac{\partial S^*}{\partial p} < 0$ (from Proposition 1) imply $\frac{dR^*}{dp} = \frac{\partial R}{\partial S}\frac{\partial S^*}{\partial p} < 0$. Perhaps surprisingly we can give a definite answer to this question when the monopolist optimally chooses both p^* and r^*. Lemma 2 characterizes the profit-maximizing choice of the referral fee, and Proposition 2 evaluates the price effects of consumer referrals.

Lemma 2. *The optimal referral fee for monopoly is $r^* = r^*(p, \pi)$ that satisfies*

$$R^*(p, r^*, \pi) = (p - r^*)\frac{dR^*(p, r^*, \pi)}{dr}. \tag{7}$$

On the one hand, a higher referral fee that the seller pays to referrers would directly reduce it profits from referrals by imposing a higher marginal cost. On the hand, a higher referral fee increases referrals by $\frac{dR^*}{dr} > 0$. The optimal referral fee is set so that the marginal cost R^* (per uninformed consumer willing to buy the product) equals the marginal benefit $(p-r)\frac{dR^*}{dr}$. Lemma 2 holds for any $\pi \in (0, 1)$ and $p \geq 0$, as long as referrals are supported by monopoly in the referral equilibrium. For instance, this would require that $p > r > r_0$.

Next we show that the monopolist would choose not to distort its pricing after the introduction of active consumer referrals.

Proposition 2. *The optimal monopoly policy under consumer referrals involves setting the standard monopoly price p^m.*

The price effects of consumer referrals could potentially be ambiguous. On the one hand, referral fees raise the cost of selling the product for the monopolist, and we would expect monopoly to have a higher price under referrals. On the other hand, a higher price means less referral incentives. Monopoly should set a lower price to support a higher equilibrium referral reach. The two effects cancel each other when monopoly is optimally choosing its price and referral fee, and consumer behavior is described by the referral equilibrium.

Lemma 3 describes an interesting property of the consumer referral equilibrium under an optimally chosen referral fee.

Lemma 3. *Monopoly profit-maximizing referral fee $r^* = r^*(p, \pi)$ is such that in the consumer referral equilibrium the expected benefit to the monopoly from an additional referral equals the cost of referral to consumers:*

$$p(1 - G(p))(1 - \pi)(1 - R^*(p, r^*, \pi)) = c. \tag{8}$$

The interpretation of the above lemma is clear. The optimal referral fee leaves no rent from referrals.

Next we compare the extent of monopoly advertising under no referrals and under the optimal referral policy. Define $\overline{\pi}$ to be the profit-maximizing level of advertising under no referrals, $\overline{\pi} \equiv \arg\max\left(Np(1 - G\left(p\right))\pi - C(\pi)\right)$. Assuming there exists a unique $\overline{\pi} \in (0, 1)$, it is described by the first-order condition $\frac{\partial \Pi}{\partial \pi} = Np(1 - G\left(p\right)) - C'(\pi) = 0$. Proposition 3 shows that consumers referrals reduce monopoly advertising expenditures, $\pi^* < \overline{\pi}$.

Proposition 3. *Monopolist advertises less when it supports active consumer referrals than under no referrals, and the result is true for any price and referral fee consistent with active referrals.*

There are two reasons why monopolist would cut on advertising expenditures. First, less advertising means more uninformed consumers who can potentially become informed through referrals. Second, more uninformed consumers implies higher referral incentives and a higher proportion of consumers receiving referrals. Proposition 3 holds for any p and r consistent with active referrals.

Assuming that monopoly selects an optimal referral fee, we can further describe the optimal advertisement level as follows.

Proposition 3'. *Conditional on the monopoly choosing the optimal referral fee that sustains active consumer referrals, the optimal monopoly level of advertising is π^* that satisfies*

$$\frac{cN}{1 - \pi^*} = C'(\pi^*). \tag{9}$$

The proof of Proposition 3' makes use of the optimality conditions for the monopoly choice of advertising intensity and referral fees. It holds for any price that is consistent with consumer referrals. Since by Lemma 3, the optimal referral fee is such that $\frac{cN}{1-\pi^*} = Np(1 - G(p))(1 - R^*(p, r^*, \pi))$, the result can be interpreted as follows. Monopoly chooses an advertising level to equalize the marginal cost of advertising to the marginal benefit of reaching consumers through ads.

Proposition 4 provides a sufficient condition for the monopolist to use active consumer referrals.

Proposition 4. *The optimal policy of monopoly involves an actively used referral policy if the referral cost is sufficiently small,*

$$c < (1-\overline{\pi})p^m(1-G(p^m)). \tag{10}$$

Intuitively, monopoly will support active consumer referrals when referral costs are sufficiently small. The condition on the referral cost can be written as $p^m > \frac{c}{(1-\overline{\pi})(1-G(p^m))} > \frac{c}{(1-\pi)(1-G(p^m))}$. When this inequality holds, monopoly can set a referral fee slightly below the monopoly price to support consumers referrals and earn extra profits from referral consumers.

We finish this sections by describing some comparative statics results for the optimal monopoly policy. Lemma 4 states what happens when the referral cost increases.

Lemma 4. *When advertising intensity is exogenously fixed and the cost of making referrals increases, monopoly raises the referral fee and keeps its price unchanged. When advertising intensity is endogenous, a higher referral cost results in a higher referral fee and less advertising, as long as the cost of advertising is not too convex (less convex than Butters' technology); a referral cost does not affect the price.*

Intuitively, monopoly usually chooses to provide stronger referral incentives when it is costly for consumers to make referrals. On the other hand, when it is not costly for consumers to refer their contacts, more of them attempt to make referrals, resulting in a higher level of congestion in referrals. Monopoly responds to this by lowering incentives for referrals - it reduces the referral fee and increases its advertising level, leaving fewer consumers uninformed.

4 The Social Welfare

Let us examine the impact of consumer referrals on the social welfare defined as the sum of monopoly profits and consumer surplus, $W = \Pi + CS$. Monopoly profits can be written as

$$\begin{aligned}\Pi(p,r,\pi) &= N\left[\pi\int_p^1 pg(v)dv + (1-\pi)\,R^*(\pi,r,p)\int_p^1 (p-r)\,g(v)dv\right] - C(\pi) \qquad (11)\\ &= N\left[\pi + (1-\pi)\,R^*(\pi,r,p)\right]\int_p^1 pg(v)dv - N\,(1-\pi)\,R^*(\pi,r,p)(1-G\,(p))r - C(\pi).\end{aligned}$$

and consumer surplus is

$$CS(p,r,\pi) = N\left[\pi + (1-\pi)\,R^*(\pi,r,p)\right]\int_p^1 (v-p)g(v)dv. \qquad (12)$$

Consumer surplus is the expected net benefit of a consumer, $\int_p^1 (v-p)g(v)dv$, times the measure of consumers informed through advertising or consumer referrals, $N\left[\pi + (1-\pi)\,R^*(\pi,r,p)\right]$. The net benefit to consumers from making referrals is zero. Consumers prefer that monopoly makes as many consumers informed (directly or indirectly) as possible. Consumers also prefer as low price as possible and as high a referral fee as possible.

The social welfare is

$$W = N\left[\pi + (1-\pi)\,R^*(\pi,r,p)\right]\left(\int_p^1 vg(v)dv\right) - NS^*(\pi,r,p)c - C(\pi).$$

because from the consumer referral equilibrium condition, $S^*c = (1-\pi)(1-G\,(p))rR^*$ (that is, in the referral equilibrium, the total cost of sending referrals equals the total benefits to consumers from sending them). The social welfare is the value of the product to consumers net of referral and advertising costs.

Proposition 5 shows that allowing monopoly to support consumer referrals generally results in a Pareto improvement.

Proposition 5. *The equilibrium allocation achieved under the optimal monopoly policy that supports consumer referrals Pareto-dominates the one achieved when monopoly cannot use consumer referrals.*

Proof. The monopolist cannot be worse-off if it chooses to support referrals. Consumer demand expands and the monopoly price is unchanged under referrals. Hence, both the monopolist and consumers benefit from the presence of consumer referrals.$\blacksquare$

It follows that if monopoly supports consumer referrals, it is socially optimal to do so. Is it true that monopoly some times does not support referrals although it is socially optimal to do so, or is monopoly supporting active referrals if and only if it is socially desirable? When is it socially optimal to support consumer referrals? Proposition 5'.provides a sufficient condition for social desirability of consumer referrals.

Proposition 5'. *For any (p, r, π) such that $\pi \in (0, 1)$ and $p > r > r_0$ it is socially optimal to support consumer referrals.*

The following proposition characterizes the socially optimal price, level of advertising, and referral fees.

Proposition 6. *The socially optimal policy (p^W, r^W, π^W) is characterized by (i) $p^W = 0$, (ii) π^W such that $\frac{cN}{1-\pi^W} = C'(\pi^W)$, and (iii) $r^W = \frac{S^*}{1-e^{-S^*}} \frac{c}{(1-\pi^W)}$, where $S^* = \log\left(\frac{(1-\pi^W)}{c}\bar{v}\right)$ and $\bar{v} = \int_0^1 vg(v)dv$. Comparing (p^*, r^*, π^*) and (p^W, r^W, π^W),*

(i) Monopoly charges a higher price $p = p^m$ than the socially optimal price $p^W = 0$.

(ii) Monopoly provides the socially optimal level of advertising.

(iii) Monopoly supports a lower referral intensity S^* than is social optimal, both conditionally on price and advertising levels and unconditionally.

Monopoly price does not change in the presence of referrals. Intuitively, monopoly does not sacrifice any of the profits it earns from the informed consumers to attract more of the uninformed consumers. It uses instruments other than price to provide consumers with adequate incentives for referring their contacts. Statement (i) is the usual result of social inefficiency due to monopoly power. Perhaps surprisingly, monopoly chooses the socially optimal advertising level but supports fewer referrals than is socially desirable.

5 Related Literature

The paper relates to several lines of research. First, there is an extensive literature on advertising. Both empirical and theoretical studies are usually based on the model of random advertising (Butters, 1977). There is also a great number of recent papers that study social networks and their impact on social learning, search, and word of mouth. Word-of-mouth consumer communication has been examined in the context of the labor markets (Calvó-Armengol and Zenou, 2005) and in the social learning environment by Ellison and Fudenberg (1995).

In marketing literature, consumer WOM has been researched in several papers, but only few look into the interaction of pricing and referral bonuses and study welfare implications. Godes and Mayzlin (2004) discuss the benefits of supporting word-of-mouth behavior by consumers as an alternative to mass advertising.

In Kornish and Li (2010), consumers care about each other, and this affects their referral decisions. Each of the existing customers have a fixed number of friends they could refer to monopoly. They are better informed about their friend's (average) value for the product than the friend. The existing customer (Sender) decides whether to refer his friend and the friend, in turn, decides whether to follow the referral. If the friend buys, she receives a utility of $v - p$, where v is the random variable. Upon the purchase, the sender receives a referral bonus B from the firm and also obtains a benefit that depends on the expected utility of the friend. The equilibrium is characterized by a threshhold value such that referrals are given if and only if the product value signal is sufficiently strong, and all recommendations are followed. This paper is interesting because it brings about the moral hazard in referrals. The bonus not just increases the awareness about the product but creates incentives for existing consumer to over-sell the product.

In a recent paper, Jun and Kim (2008) propose a model in which referrals are spread along a chain of consumers and a monopoly chooses a price and referral fee. More specifically, n consumers seek to buy at most one unit of a good produced by a monopoly at a constant marginal cost $c \geq 0$. Consumer values v_i for the product are independently and identically distributed according to a known distribution function F. Information about the product flows along a simple chain connecting each consumer to the next one, from consumer 1 to consumer n. One consumer is exogenously informed about the product and can pass it to one other uninformed consumer, who can pass it on. Only a consumer who buys the product can refer the next-in-line consumer, and this is called buy-to-refer (BTR) constraint. The cost to consumer of making a referral $\varrho \geq 0$ is constant, and it must be incurred even if the referral does not lead to a purchase. Monopoly chooses a price $p \geq 0$ and referral fee $r \geq 0$ (per unit of product sold), and then consumers decide on their buying and referral strategies.

Because of the specific nature of the consumer network, questions about referral congestion, referral intensity, and targeting of referral messages cannot be effectively addressed in the model. It would be fruitful to consider a more general network with many-to-many consumer connections. In such a model, monopoly would choose a level of random advertising and referral incentives and consumers would choose a number of referrals to make (referral intensity) and whom to target with referral messages. This would be a step forward that brings together the growing research on social networks and a more traditional literature on information transmission in consumer markets.

6 Conclusion

A variety of information channels are available to sellers who market their products to consumers. These include traditional mass advertisements on TV and in newspapers, targeted promotional advertising, as well as buzz marketing, consumer word of mouth (WOM), and consumer referral policies. We look at the optimal marketing mix between advertising, referral policy, and price promotion, and discuss the welfare impacts of referrals. We allow for the possibility of congestion in both advertising and referral messages.[5]

[5] Anderson and de Palma (2009) introduce a model of advertising with information congestion.

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Appendix A: Proofs

Proposition 1. Proof. Consider the informed consumers' equilibrium choice of referral intensity q. First, suppose that there are $n = (1 - G(p))\pi N > 0$ informed consumers who purchase the product and $m = (1-\pi)N > 0$ uninformed consumers, where n is large. Referral attempts are made randomly. With probability $1 - \pi$ referral attempts reach the uninformed consumers. We assume that if a consumer receives referral attempts from k informed type consumers, then she chooses one with equal probability $1/k$. Focusing on a symmetric equilibrium, suppose that $n - 1$ informed consumers are choosing q, while the remaining informed consumer i chooses q_i. Then, the proportion of the uninformed consumers who use referrals from i is

$$F_i(q_i, q) = \sum_{k=0}^{n-1} \frac{1}{k+1} q_i (1-q)^{n-1-k} q^k \times C(n-1, k), \tag{13}$$

where $C(n-1, k) = (n-1)!/(n-1-k)!k!$. Note that the term $(1-q)^{n-1-k}q^k \times C(n-1, k)$ denotes the probability that an uninformed consumer receives k referral contacts from other $n-1$ referrers. By rearranging the formula, we obtain,

$$\begin{aligned} F_i(q_i, q) &= \sum_{k=1}^{n} \frac{1}{k} q_i (1-q)^{n-k} q^{k-1} \times C(n-1, k-1) \qquad (14) \\ &= \sum_{k=1}^{n} \frac{1}{k} q_i (1-q)^{n-k} q^{k-1} \times \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{1}{n} \sum_{k=1}^{n} q_i (1-q)^{n-k} q^{k-1} \times \frac{n!}{k!(n-k)!} \\ &= \frac{1}{n} \times \frac{q_i}{q} \sum_{k=1}^{n} (1-q)^{n-k} q^k \times C(n, k) \\ &= \frac{q_i}{nq} \left[1 - (1-q)^n\right], \end{aligned}$$

for $q > 0$, and $F_i(q_i, 0) = q_i$ for $q = 0$. Referrer i's optimal referral choice q_i is obtained by solving

$$\max_{q_i} \left((1 - G(p)) \times r \times m \times \frac{q_i}{nq} \left[1 - (1-q)^n\right] - c q_i N \right) \tag{15}$$

for $q \in (0,1]$ and

$$\max_{q_i} \left(\left(1 - G\left(p\right)\right) \times r \times m \times q_i - cq_i N \right) \tag{16}$$

for $q = 0$, where the term $1 - G(p)$ denotes the probability that an uninformed consumer purchases the product and cq_iN denotes the cost of referring a fraction q_i of all consumers. Rewriting the problem using $m = (1-\pi)N$, we obtain

$$\max_{q_i} q_i N \left(\left(1 - G\left(p\right)\right) (1-\pi) r \frac{\left[1 - (1-q)^n\right]}{nq} - c \right) \tag{17}$$

for $q \in (0,1]$ and

$$\max_{q_i} q_i N \left(\left(1 - G\left(p\right)\right) (1-\pi) r - c \right) \tag{18}$$

for $q = 0$.

Referrer i's objective function is linear in q_i. Note that $\left[1 - (1-q)^n\right]/nq < 1$ for $q \in (0,1]$ and $n \geq 2$.[6] For $r \leq \frac{c}{(1-\pi)(1-G(p))}$ and $q \in (0,1]$, the unique best response is $q_i = 0$. Thus, in this case, $q^* \in (0,1]$ cannot be the equilibrium referral intensity in a symmetric equilibrium, and the only symmetric equilibrium is $q^* = 0$.

In a symmetric interior equilibrium, consumers are indifferent among all q_is. The symmetric equilibrium q^* is implicitly calculated as a unique solution to

$$\frac{q^* n}{1 - e^{-q^* n}} = \frac{(1-\pi)(1-G\left(p\right))r}{c}. \tag{19}$$

where $c > 0$, $\pi > 0$, $p \geq 0$, and $r \geq 0$. We used the approximation $1 - (1-q^*)^n \simeq 1 - e^{-q^* n}$ for small q^* because $\log(1-q)^n = n\log(1-q) \simeq -nq$, and $(1-q)^n \simeq e^{-nq}$ for small q.

When $r > \frac{c}{(1-\pi)(1-G(p))}$, the only candidate for the symmetric equilibrium intensity is $S^* = q^* n$ that solves (*). The LHS of (*) is not less than unity, and therefore this equation has solution only if $c < (1-\pi)(1-G\left(p\right))r$, or, in terms

[6] We can show by induction that $(1-q)^n > 1 - nq$ for $q \in (0,1]$ and $n \geq 2$. When $n = 2$, $(1-q)^2 = 1 - 2q + q^2 > 1 - 2q$, and the result is true for $n = 2$. Suppose it is true for some n, i.e. $(1-q)^n > 1 - nq$. We need to show that it is then true for $n+1$, i.e. $(1-q)^{n+1} > 1 - (n+1)q$
note that $(1-q)^{n+1} = (1-q)(1-q)^n > (1-q)(1-nq)$ by the inductive hypothesis, and therefore $(1-q)^{n+1} > 1 - (n+1)q + nq^2 > 1 - (n+1)q$.

of referral fee, $r > \frac{c}{(1-\pi)(1-G(p))}$. Then, indeed, given that others are choosing q^*, consumer i obtains a zero payoff for any strategy, so she might as well choose q^*. Thus, q^* is the symmetric referral equilibrium and $S^* > 0$ is the equilibrium referral intensity when $r > \frac{c}{(1-\pi)(1-G(p))}$. Note that the LHS of (*) is a strictly increasing function of $S^* = q^* n$, where $n = (1 - G(p))\pi N$. This implies that the equilibrium referral intensity S^* is unique when it exists. The RHS of (*) is increasing in r and decreasing in π, p, and c. Hence, the equilibrium referral intensity, $S^* = S^*(p, r, \pi) = q^* n$, is increasing in r and decreasing in π, p, and c: $\frac{\partial S^*}{\partial r} > 0$, $\frac{\partial S^*}{\partial \pi} < 0$, $\frac{\partial S^*}{\partial p} < 0$, and $\frac{\partial S^*}{\partial c} < 0$ for $r > \frac{c}{(1-\pi)(1-G(p))}$.■

Lemma 1. Proof. The referral equilibrium is described by $\varphi(S^*) = \frac{b}{c}$, where $\varphi(S) = \frac{S}{1-e^{-S}}$ and $b = (1-\pi)(1-G(p))r$. Let us find how the equilibrium referral reach depends on p and r: $\frac{dR^*}{dp} = \frac{\partial R}{\partial S}\frac{\partial S^*}{\partial p}$ and $\frac{dR^*}{dr} = \frac{\partial R}{\partial S}\frac{\partial S^*}{\partial r}$. It follows that $\frac{dR^*}{dp} / \frac{dR^*}{dr} = \frac{\partial S^*}{\partial p} / \frac{\partial S^*}{\partial r}$. Totally differentiating the equilibrium referral condition $\varphi(S^*(p, r, \pi)) = \frac{b}{c}$ with respect to p, we obtain $\varphi'(S^*)\frac{\partial S^*}{\partial p} = \frac{1}{c}\frac{\partial b}{\partial p}$. Similarly, differentiating with respect to r, we obtain $\varphi'(S^*)\frac{\partial S^*}{\partial r} = \frac{1}{c}\frac{\partial b}{\partial r}$. Hence, $\frac{\partial S^*}{\partial p} / \frac{\partial S^*}{\partial r} = \frac{\partial b}{\partial p} / \frac{\partial b}{\partial r}$. Finally, note that $\frac{\partial b}{\partial p} = -(1-\pi)g(p)\, r < 0$ and $\frac{\partial b}{\partial r} = (1-\pi)(1-G(p)) > 0$. We conclude that $\frac{dR^*}{dp} / \frac{dR^*}{dr} = \frac{\partial S^*}{\partial p} / \frac{\partial S^*}{\partial r} = \frac{\partial b}{\partial p} / \frac{\partial b}{\partial r} = -hr$, where $h = h(p) = \frac{g(p)}{1-G(p)}$ is the hazard function.

Similarly, we have $\frac{dR^*}{d\pi} / \frac{dR^*}{dr} = \frac{\partial S^*}{\partial \pi} / \frac{\partial S^*}{\partial r} = \frac{\partial b}{\partial \pi} / \frac{\partial b}{\partial r} = -\frac{r}{1-\pi}$ because $\frac{\partial b}{\partial \pi} = -(1-G(p))r$.■

Lemma 2. Proof. With respect to referral fee, r, the f.o.c. is

$$\frac{1}{N}\frac{\partial \Pi}{\partial r} = -(1-G(p))(1-\pi)R^* + (p-r)(1-G(p))(1-\pi)\frac{dR^*}{dr} = 0. \qquad (20)$$

It follows that an interior optimal referral fee r must satisfy $R^* + (p-r)\frac{dR^*}{dr} = 0$.■

Proposition 2. Proof. With respect to price, p, the f.o.c. for profit maximization is

$$\begin{aligned}\frac{1}{N}\frac{\partial\Pi}{\partial p} &= \frac{\partial\left(p(1-G\left(p\right))\right)}{\partial p}\pi \\ &+\frac{\partial\left((p-r)(1-G\left(p\right))\right)}{\partial p}(1-\pi)R^{*} \\ &+(p-r)(1-G\left(p\right))(1-\pi)\frac{dR^{*}}{dp}\end{aligned} \tag{21}$$

Active referrals imply that $\pi\in(0,1)$, $r>0$ and $R^{*}>0$. Note that the second term of the RHS is

$$\frac{\partial\left((p-r)(1-G\left(p\right))\right)}{\partial p}(1-\pi)R^{*}=\frac{\partial\left(p(1-G\left(p\right)\right)}{\partial p}(1-\pi)R^{*}-\frac{\partial r(1-G\left(p\right))}{\partial p}(1-\pi)R^{*}$$

At $p=p^{m}$, $\frac{\partial(p(1-G(p)))}{\partial p}=0$ holds, and we have

$$\begin{aligned}\frac{1}{N}\frac{\partial\Pi}{\partial p}|_{p=p^{m}} &= rg\left(p^{m}\right)(1-\pi)R^{*} \\ &+(p^{m}-r)(1-G\left(p^{m}\right))(1-\pi)\frac{dR^{*}}{dp}\end{aligned} \tag{22}$$

Since $\frac{dR^{*}}{dp}<0$, the price effects of consumer referrals could potentially be ambiguous. However, we can show that an optimally chosen referral fee, monopoly does not have an incentive to change its price from the standard monopoly level, p^{m}. From Lemma 1, $\frac{dR^{*}}{dp}/\frac{dR^{*}}{dr}=-hr$, and from Lemma 2, the condition for the optimal referral fee is $(p^{m}-r)=R^{*}/\left(\frac{dR^{*}}{dr}\right)$. Substituting these conditions into the expression for $\frac{1}{N}\frac{\partial\Pi}{\partial p}|_{p=p^{m}}$, we have

$$\begin{aligned}\frac{1}{N}\frac{\partial\Pi}{\partial p}|_{p=p^{m}} &= rg\left(p^{m}\right)(1-\pi)R^{*}+(1-G\left(p^{m}\right))(1-\pi)R^{*}\frac{dR^{*}}{dp}/\frac{dR^{*}}{dr} \\ &= R^{*}\left[rg\left(p^{m}\right)(1-\pi)-(1-G\left(p^{m}\right))(1-\pi)h(p^{m})r\right] \\ &= 0.\end{aligned} \tag{23}$$

where the last equality holds since $h(p)=\frac{g(p)}{1-G(p)}$. Therefore, the profit-maximizing price under referrals is equal to p^{m}.■

Lemma 3. Proof. To find the optimal r, given p and π, let $B\equiv\frac{(1-\pi)(1-G(p))}{c}$. The equilibrium S^{*} is the solution to

$$\varphi(S^{*})=\frac{S^{*}}{1-e^{-S^{*}}}=Br, \tag{24}$$

which can be written in terms of r as

$$r = \frac{S^*}{1 - e^{-S^*}} \frac{1}{B}. \tag{25}$$

Totally differentiating the equilibrium condition for S^* with respect to r, we find that $\varphi'(S^*)\frac{\partial S^*}{\partial r} = B$, or $\frac{\partial S^*}{\partial r} = \frac{B}{\varphi'(S^*)}$, where $\varphi'(S^*) = \dfrac{1 - e^{-S^*} - S^* e^{-S^*}}{(1 - e^{-S^*})^2}$. From Lemma 2, the optimal referral fee satisfies $R^* = (p - r)\frac{\partial R}{\partial S}\frac{\partial S^*}{\partial r}$. This condition can be written as

$$1 - e^{-S^*} = (p - r)e^{-S^*} \frac{B}{\varphi'(S^*)}, \tag{26}$$

since $R^* = R(S^*) = 1 - e^{-S^*} > 0$ and $\frac{\partial R}{\partial S} = e^{-S} > 0$. Substituting $r = \frac{S^*}{1-e^{-S^*}}\frac{1}{B}$ and simplifying the condition, we obtain $1 - pe^{-S^*}B = 0$, or

$$(1 - \pi)(1 - G(p))(1 - R^*(p, r, \pi))p = c. \tag{27}$$

■

Proposition 3. Proof. Without referrals, the monopoly profit is $\Pi(p, r, \pi) = Np(1 - G(p))\pi - C(\pi)$, and the

marginal profit with respect to the level of advertising π is $\frac{\partial \Pi}{\partial \pi} = Np(1 - G(p)) - C'(\pi)$. Assuming $C'(0) < Np(1 - G(p))$, $C'(\pi) > 0$ and $C''(\pi) \geq 0$, there exists a unique profit-maximizing level of advertising under no referrals: $\overline{\pi} = C'^{-1}(Np(1 - G(p)))$.

With referrals, the monopoly profit is

$$\Pi(p, r, \pi) = Np(1 - G(p))\pi + N(p - r)(1 - G(p))(1 - \pi)R^*(\pi, r, p) - C(\pi), \tag{28}$$

and the marginal profit of advertising is

$$\begin{aligned} \frac{\partial \Pi}{\partial \pi} &= Np(1 - G(p)) - C'(\pi) \\ &\quad + N(p - r)(1 - G(p))\left(-R^* + (1 - \pi)\frac{dR^*}{d\pi}\right). \end{aligned} \tag{29}$$

The first line in the expression is the same as under no referrals, and the second line is negative since $R^* > 0$, $\frac{dR^*}{d\pi} = \frac{\partial R}{\partial S}\frac{\partial S^*}{\partial \pi}$, $\frac{dR^*}{dS} > 0$, and $\frac{\partial S^*}{\partial \pi} < 0$ under consumer referrals. Hence, the monopoly would advertise less, $\pi^* < \overline{\pi}$, when it supports referrals, and the result holds for any p and r consistent with active referrals (a sufficient condition for that $p > r > \frac{c}{(1-\overline{\pi})(1-G(p))}$).■

Proposition 3'. Proof. With referrals, the monopoly profit is $\Pi(p,r,\pi) = Np(1-G(p))\pi + N(p-r)(1-G(p))(1-\pi)R^*(\pi,r,p) - C(\pi)$

The marginal profit of advertising is

$$\begin{aligned}\frac{\partial \Pi}{\partial \pi} &= Np(1-G(p)) + N(p-r)(1-G(p))\left(-R^* + (1-\pi)\frac{dR^*}{d\pi}\right) - C'(\pi) \\ &= Np(1-G(p))(1-R^*) + N(1-G(p))\left[rR^* + (p-r)(1-\pi)\frac{dR^*}{d\pi}\right] - C'(\pi)\end{aligned} \quad (30)$$

The expression in the bracket is zero:

$$\begin{aligned} rR^* + (p-r)(1-\pi)\frac{dR^*}{d\pi} &= rR^* + R^*(1-\pi)\frac{dR^*}{d\pi} / \frac{dR^*}{dr} \\ &= R^*\left[r - (1-\pi) \times \frac{r}{1-\pi}\right] \\ &= 0, \end{aligned} \quad (31)$$

because by Lemma 2, $(p-r) = R^* / \left(\frac{dR^*}{dr}\right)$ and by Lemma 1, $\frac{dR^*}{d\pi} / \frac{dR^*}{dr} = -\frac{r}{1-\pi}$. Therefore, the first-order condition is

$$\frac{\partial \Pi}{\partial \pi} = Np(1-G(p))(1-R^*) - C'(\pi) = 0. \quad (32)$$

By Lemma 3,

$$(1-\pi)(1-G(p))(1-R^*(p,r^*,\pi))p = c$$

at the consumer referral equilibrium when monopoly sets the optimal referral fee $r^* = r^*(p,\pi)$. We conclude that, conditional on the monopoly choosing the optimal referral fee, the optimal monopoly level of advertising is π^* that satisfies $\frac{cN}{1-\pi^*} = C'(\pi^*)$.■

Proposition 4. Proof. From Proposition 3 it follows that $\pi^* < \overline{\pi}$, and therefore the fraction of the uninformed is higher under referrals $1-\pi^* > 1-\overline{\pi}$. By

Proposition 2, monopoly chooses price p^m regardless of whether referrals are present. Suppose referral cost is sufficiently low: $c < (1-\overline{\pi})p^m(1-G(p^m))$. Then, there exists $r \in (\frac{c}{(1-\overline{\pi})(1-G(p^m))}, p^m)$, and from Proposition 1, active referrals are supported for such r: $S^* > 0$ and monopoly reaches a positive measure of uninformed consumers through referrals, $R^* = R(S^*) = 1 - e^{-S^*} > 0$. Monopoly receives positive additional profits from the referral consumers, without altering its profits from the informed consumers. Thus, when referral cost is sufficiently low, the monopoly profits improve by introducing a referral policy.■

Lemma 4. Proof. From Lemma 3, the equilibrium referral reach for an optimally chosen referral fee is

$$R^*(p, r^*(p, \pi; c), \pi) = 1 - \frac{c}{(1-\pi)p(1-G(p))} > 0$$

when $p(1-G(p))(1-\pi) > c$. Since $S^* = -\ln(1-R^*)$, it follows that

$$\begin{aligned} S^* &= \ln\left(\frac{p(1-G(p))(1-\pi)}{c}\right) \\ &= \ln p(1-G(p)) + \ln(1-\pi) - \ln(c), \end{aligned}$$

and $\frac{\partial S^*}{\partial c} = -\frac{1}{c} < 0$ and $\frac{\partial S^*}{\partial \pi} = -\frac{1}{1-\pi} < 0$.

From Proposition 1, in the referral equilibrium, $\varphi(S^*) = \frac{(1-\pi)(1-G(p))}{c}r$. Hence, for given π and p, The optimal referral fee can be written as

$$r^* = \varphi(S^*)\frac{c}{(1-\pi)(1-G(p))}$$

where $\varphi(S^*) = \dfrac{S^*}{1-e^{-S^*}}$. Rewriting the expression for r^*, we obtain

$$r^* = p\frac{\ln\left(\frac{p(1-G(p))(1-\pi)}{c}\right)}{\frac{p(1-G(p))(1-\pi)}{c} - 1}.$$

To see that $r^* = r^*(p, \pi; c)$ is an increasing function of c and π, let $x \equiv \frac{p(1-G(p))(1-\pi)}{c}$; then, $r^* = p\frac{\ln(x)}{x-1}$, and $\frac{\ln(x)}{x-1}$ is a strictly decreasing function of x because $\frac{\partial}{\partial x}\left(\frac{\ln(x)}{x-1}\right) = \left(1 - \frac{1}{x} - \ln x\right)(1-x)^{-2} < 0$ for $x > 0$.

Next, assume that monopoly could adjust price and advertising in response to chances in referral fees. From Proposition 2, monopoly sets price p^m irrespective of the levels of advertising and referral fees.

From Proposition 3', the optimal referral fee and advertising are such that $cN = C'(\pi^*)(1-\pi^*)$ holds. Differentiating with respect to c, we obtain

$$N = (C''(\pi)(1-\pi^*) - C'(\pi^*)) \frac{\partial \pi^*}{\partial c},$$

or

$$\frac{\partial \pi^*}{\partial c} = \frac{N}{C''(\pi^*)(1-\pi^*) - C'(\pi^*)},$$

assuming $C''(\pi^*)(1-\pi^*) - C'(\pi^*) \neq 0$. Since $C'(\pi) > 0$ and $C''(\pi) \geq 0$, the effect of referral fee on advertising intensity is ambiguous. However, if the advertising cost less convex than the standard Butters' technology $C(\pi) = a \ln\left(\frac{1}{1-\pi}\right)$, for which $C''(\pi)(1-\pi) - C'(\pi) = 0$, an increase in the referral cost would reduce the monopoly advertising intensity. For example, for a linear function $C(\pi) = a\pi$, $C''(\pi) = 0$, and $\frac{\partial \pi^*}{\partial c} = -\frac{N}{C'(\pi^*)} < 0$.

We can now evaluate the total effect of a change in referral cost on referral fees. Since $r^* = p\frac{\ln(x)}{x-1}$ is decreasing in $x = \frac{p(1-G(p))(1-\pi^*)}{c}$ and the price is independent of c, we only need to evaluate the sign of $\dfrac{d((1-\pi^*(c))/c)}{dc} = -\frac{\partial \pi^*}{\partial c}\frac{1}{c} - \frac{(1-\pi^*)}{c^2}$. Plugging in the expression for $\frac{\partial \pi^*}{\partial c}$ and recognizing that by Proposition 3' monopoly sets $cN = C'(\pi^*)(1-\pi^*)$, we find that

$$-\frac{d\left(\frac{(1-\pi^*(c))}{c}\right)}{dc}\frac{c^2}{(1-\pi^*)} = \frac{C''(\pi^*)(1-\pi^*)}{C''(\pi^*)(1-\pi^*) - C'(\pi^*)} \leq 0$$

iff $C''(\pi)(1-\pi) - C'(\pi) < 0$. That is, the cost of advertising function is less convex than the standard Butters' technology $C(\pi) = a \ln\left(\frac{1}{1-\pi}\right)$, for which $\frac{\partial r^*}{\partial c} = 0$. For a linear advertising cost function $C(\pi) = A\pi$, $C''(\pi) = 0$, and therefore, $\frac{\partial r^*}{\partial c} > 0$. To conclude, we find that the optimal referral fee is increasing in the referral cost, $\frac{dr^*}{dc} > 0$ as long as advertising cost function is not too convex.■

Proof of Proposition 5'. The social welfare in the absence of referrals is $W = N\pi\left(\int_p^1 vg(v)dv\right) - C(\pi)$. Hence, the sufficient condition for referrals to be socially beneficial is

$$(1-\pi)\, R^{*}\left(\int_{p}^{1} v g(v) dv\right) > (1-\pi)\,(1-G\,(p)) R^{*} r.$$

For $1-\pi > 0$ and $R^{*} > 0$, this inequality is equivalent to $\left(\int_{p}^{1} v g(v) dv\right) - (1-G\,(p)) r > 0$, which holds since $\int_{p}^{1} v g(v) dv \geq \int_{p}^{1} p g(v) dv \geq p(1-G\,(p)) > r(1-G\,(p))$.■

Proposition 6. Proof. The first order conditions with respect to p, r and π are

$$\begin{aligned} \frac{1}{N}\frac{\partial W}{\partial p} &= (1-\pi)\frac{dR^{*}}{dp}\int_{p}^{1} v g(v) dv - c\frac{dS^{*}}{dp} - [\pi + (1-\pi)\, R^{*}(\pi, r, p)]\, p g(p) \qquad (33) \\ &= \left[(1-\pi)\, e^{-S^{*}}\int_{p}^{1} v g(v) dv - c\right]\frac{dS^{*}}{dp} - [\pi + (1-\pi)\, R^{*}(\pi, r, p)]\, p g(p) = 0 \end{aligned}$$

$$\begin{aligned} \frac{1}{N}\frac{\partial W}{\partial r} &= (1-\pi)\frac{dR^{*}}{dr}\int_{p}^{1} v g(v) dv - c\frac{dS^{*}}{dr} \qquad (34) \\ &= \left[(1-\pi)\, e^{-S^{*}}\int_{p}^{1} v g(v) dv - c\right]\frac{dS^{*}}{dr} = 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial W}{\partial \pi} &= N\left[(1-R^{*})\int_{p}^{1} v g(v) dv + (1-\pi)\frac{dR^{*}}{d\pi}\int_{p}^{1} v g(v) dv - c\frac{dS^{*}}{d\pi}\right] - C'(\pi) (35) \\ &= N\left[(1-R^{*})\int_{p}^{1} v g(v) dv + \left\{(1-\pi)\, e^{-S^{*}}\int_{p}^{1} v g(v) dv - c\right\}\frac{dS^{*}}{d\pi}\right] - C'(\pi) \end{aligned}$$

respectively, where $\frac{dR^{*}}{dx} = \frac{\partial R}{\partial S}\frac{dS^{*}}{dx} = e^{-S^{*}} \times \frac{dS^{*}}{dx}$ for $x = p, r,$ and π. By Proposition 1, $\frac{dS^{*}}{dp} < 0$, $\frac{dS^{*}}{dr} > 0$, and $\frac{dS^{*}}{d\pi} < 0$.

From the f.o.c. with respect to r, we obtain

$$(1-\pi)\, e^{-S^{*}}\int_{p}^{1} v g(v) dv - c = 0, \qquad (36)$$

and therefore r^{W} is such that $S^{*} = \log\left(\frac{(1-\pi)}{c}\int_{p}^{1} v g(v) dv\right)$.

Substituting $(1-\pi)\, e^{-S^{*}}\int_{p}^{1} v g(v) dv - c = 0$ into the f.o.c. with respect to p, we obtain $p^{W} = 0$. That is, the social welfare maximizing price is zero. Substituting these conditions into the f.o.c. with respect to π, we obtain

$$N\,(1-R^{*})\,\bar{v} - C'(\pi) = 0. \qquad (37)$$

Since $1 - R^* = e^{-S^*}$, the f.o.c. with respect to r with $p = 0$ generates

$$(1-\pi)(1-R^*)\bar{v} - c = 0. \tag{38}$$

Putting the last two equations together, we find that $\frac{cN}{1-\pi^W} = C'(\pi^W)$ holds at the socially optimal level of advertising π^W. This completes the proof.■

Proposition 6' Proof. We only need to show (iii). Lemma 3 shows that monopoly would choose $r = r^m$ such that the equilibrium referral reach $R^* = 1 - e^{-S^*}$ satisfies

$$(1-\pi)(1-G(p))(1-R^*)p = c. \tag{39}$$

Rewriting this in terms of S^* we obtain

$$S^* = \log\left(\frac{(1-\pi)(1-G(p))p}{c}\right). \tag{40}$$

Evaluating S^* at the profit-maximizing levels of price and advertising, p^m and π^m, yields $S^* = \log\left(\frac{(1-\pi^m)(1-G(p^m))p^m}{c}\right)$.

According to Proposition (6), r^W is such that $S^* = \log\left(\frac{(1-\pi)}{c}\int_p^1 vg(v)dv\right)$. Evaluating S^* at the socially optimal levels of price and advertising, $p^W = 0$ and π^W, $S^* = \log\left(\frac{(1-\pi^W)}{c}\bar{v}\right)$, where $\bar{v} = \int_0^1 vg(v)dv$.

Since $\pi^W = \pi^m$ and $(1-G(p))p = \int_p^1 pg(v)dv < \int_p^1 vg(v)dv$, the socially optimal referral intensity and the referral fee are higher than the levels chosen by profit-maximizing monopoly, for any given price p. Moreover, since $p^W < p^m$, and $\frac{dS^*}{dp} > 0$ for $p < p^m$, monopoly undersupports referrals unconditionally as well.■

Appendix B. Extension to Better Informed Consumers

The monopolist's information is as before: she knows that the cumulative density function of willingness-to-pay for the general population is G. The departure from the basic model is to assume that there are two subgroups of consumers: H and L with fractions α^H and α^L, respectively ($\alpha^H + \alpha^L = 1$). We assume that consumers can tell which group each of other consumers belongs to, while the monopolist cannot distinguish these two groups of consumers. However, the exact willingness-to-pay is private information for each consumer. This can be regarded as consumers' relative informational advantage over the monopolist. Group H (L) consumers tend to have higher (lower) willingness-to-pay but with uncertainty. That is, some consumers who belong to group L have higher willingness-to-pay than some in group H. The cumulative density function for group H is described by $G^H : [0,1] \to [0,1]$, and similarly the one for group L is denoted by $G^L : [0,1] \to [0,1]$. We assume that for all v, $G^H(v) < G^L(v)$ holds (the first order stochastic dominance). Clearly, for all $v \in [0,1]$, $G(v) = \alpha G^H(v) + (1-\alpha)G^L(v)$ must hold. We assume that $-2g^t(p) - p\left(g^t(p)\right)' < 0$ for all p and $t \in \{H, L\}$.

Informed consumers (by monopolist's advertisement) can choose q_i^H and q_i^L as referral intensities for two different groups. Then, essentially the same analysis as in the basic model applies for referral equilibrium, and we find the following:

Proposition 1'. *There is a unique symmetric referral equilibrium characterized by the referral intensity for group $t = H, L$ consumers $S^{t*} = S^{t*}(p, r, \pi)$ such that (i) for all $r \le r_0^t \equiv \frac{c}{(1-\pi)(1-G^t(p))}$, $S^{t*} = 0$ holds, and (ii) for all $r > r_0$, $S^{t*} = S^{t*}(p, r, \pi)$ is implicitly defined by*

$$\varphi(S^{t*}) = \frac{(1-\pi)(1-G^t(p))r}{c}. \tag{41}$$

The equilibrium referral intensity $S^{t*} = S^{t*}(p, r, \pi)$ satisfies $\frac{\partial S^{t*}}{\partial \pi} < 0$, $\frac{\partial S^{t*}}{\partial p} < 0$, $\frac{\partial S^{t*}}{\partial r} > 0$, and $\frac{\partial S^{t*}}{\partial c} < 0$ for $r > r_0$.

Given the first-order stochastic dominance, we have $r_0^H < r_0^L$ and $S^{L*} < S^{H*}$ for all p, r and π. That is, referral congestion is more in group H than in group L, and it is possible that consumer referrals take place only for group H (if $r_0^H \le r < r_0^L$ holds). The total cost to consumers from giving referrals equals the referral fees they collect, $N\alpha^t S^{t*} c = N\alpha^t(1-\pi)(1-G^t(p))R^{t*}r$. That is, α^t cancels out, and has no effect in determining consumer referral intensity and reach. The equilibrium referral reach is $R^{t*} = R^{t*}(p, r, \pi) = R^*(S^{t*}(p, r, \pi)) = 1 - e^{-S^{t*}(p,r,\pi)}$.

Lemma 1 in the basic model extends to this setup as well.

Lemma 1'. *In the referral equilibrium, (i)* $\frac{dR^{t*}}{dp} / \frac{dR^{t*}}{dr} = -h^t r$, *where* $h^t = h^t(p) = \frac{g^t(p)}{1-G^t(p)}$ *is the hazard function, and (ii)* $\frac{dR^{t*}}{d\pi} / \frac{dR^{t*}}{dr} = -\frac{r}{1-\pi}$. *The latter result is common to both groups.*

Let us derive monopoly profits in this environment. The number of uninformed consumers in group t who receive referrals is $(1-\pi)\alpha^t R^{t*} N$. Then, the demand the firm faces is

$$\begin{aligned} D(p) &= n(p, \pi) \\ &\quad + \sum_{t \in \{L,H\}} m^t(\pi) R^{t*}(p, r, \pi)(1 - G^t(p)). \end{aligned} \tag{42}$$

and the monopolist's profit function is

$$\Pi(p, r, \pi) = pn(p, \pi) + (p - r) \sum_{t \in \{L,H\}} (1 - G^t(p)) m^t(\pi) R^{t*}(p, r, \pi) - C(\pi), \tag{43}$$

where $R^{t*} = R^{t*}(p, r, \pi) > 0$ for $r > r_0^t \equiv \frac{c}{(1-\pi)(1-G^t(p))}$ and $R^{t*} = 0$ otherwise.

Recall that the number of uninformed type t consumers is $m^t = m^t(\pi) = (1-\pi)\alpha^t N$ and the number of informed referring consumers is $n = n(p, \pi) = (1 - G(p))\pi N$ (note that it is irrelevant which group referrers belong to). Monopoly profits under referrals can be written as

$$\Pi(p, r, \pi) = N \left[p(1 - G(p))\pi + (p - r)(1 - \pi) \sum_{t \in \{L,H\}} (1 - G^t(p))\alpha^t R^{t*}(\pi, r, p) \right] - C(\pi) \tag{44}$$

When there are no referral activities, monopoly profit is $\Pi(p, r, \pi) = Np(1 - G(p))\pi - C(\pi)$. Clearly, the monopoly price $p^m \equiv \arg\max_p (p(1 - G(p))$ maximizes the first term in the brackets. However, it is not clear if the optimal price p under consumer referrals is higher or lower than p^m because $\frac{\partial R}{\partial S} > 0$ and $\frac{\partial S^*}{\partial p} < 0$ (from Proposition 1) imply $\frac{dR^*}{dp} = \frac{\partial R}{\partial S}\frac{\partial S^*}{\partial p} < 0$. We can modify Lemma 2.

Lemma 2'. *The optimal referral fee r for monopoly under consumer referrals satisfies*

$$\sum_{t\in\{L,H\}}(1-G^t(p))\alpha^t R^{t*}=(p-r)\sum_{t\in\{L,H\}}(1-G^t(p))\alpha^t\frac{dR^{t*}}{dr}. \qquad (45)$$

This is no longer a straightforward formula, since the dR^t/dr can be different for different types. Thus, unless the monopolist can offer type-dependent referral fees, the formula cannot be simplified to the one in Lemma 2. The point of consumer referrals is to utilize consumers' information advantage, it is unreasonable to assume that the monopolist can set type-dependent referral fees. By this reason, calculating the optimal monopoly price under active referrals for both groups is a difficult task in general. However, we can show that the monopolist would choose to increase its price after the introduction of active consumer referrals when only group H gets consumer referrals.

Proposition 2'. *Suppose that in equilibrium, group L consumers receive no referrals. Then, the optimal monopoly policy under consumer referrals is higher than the standard monopoly price p^m if the hazard rates satisfy the following condition: for all $p\in(0,1)$, we have*

$$\frac{g^H(p)}{1-G^H(p)}<\frac{g(p)}{1-G(p)}.$$

Proposition 2'. Proof. With respect to price, p, the f.o.c. for profit maximization is

$$\begin{aligned}\frac{1}{N}\frac{\partial\Pi}{\partial p} &= \frac{\partial\left(p(1-G(p))\right)}{\partial p}\pi \\ &\quad+\frac{\partial\left((p-r)(1-G^H(p)\right)}{\partial p}(1-\pi)\alpha^H R^{H*} \\ &\quad+(p-r)(1-G^H(p))(1-\pi)\alpha^H\frac{dR^{H*}}{dp}\end{aligned} \qquad (46)$$

Active referrals imply that $\pi \in (0,1)$, $r > 0$ and $R^{H*} > 0$ (and $r < r_0^L$). As before (in the proof of Proposition 2), at $p = p^m$, we have

$$\begin{aligned}\frac{1}{N}\frac{\partial \Pi}{\partial p}|_{p=p^m} &= \frac{\partial \left(p(1-G^H(p))\right)}{\partial p}|_{p=p^m} \times (1-\pi)\alpha^H R^{H*} \\ &\quad + rg^H(p^m)(1-\pi)\alpha^H R^{H*} \\ &\quad + (p^m - r)(1-G^H(p^m))(1-\pi)\alpha^H \frac{dR^{H*}}{dp}\end{aligned} \tag{47}$$

From Lemma 1', $\frac{dR^{H*}}{dp} / \frac{dR^{H*}}{dr} = -h^H r$, and from Lemma 2', the condition for the optimal referral fee in the absence of referrals to type L is $(p^m - r) = \left((1-G^H(p^m))\alpha^H R^{H*}\right) / \left((1-G^H(p^m))\alpha^H \frac{dR^{H*}}{dr}\right) = R^{H*} / \frac{dR^{H*}}{dr}$. Substituting these conditions into the expression for $\frac{1}{N}\frac{\partial \Pi}{\partial p}|_{p=p^m}$, we have

$$\begin{aligned}\frac{1}{N}\frac{\partial \Pi}{\partial p}|_{p=p^m} &= \frac{\partial \left(p(1-G^H(p))\right)}{\partial p}|_{p=p^m} \times (1-\pi)\alpha^H R^{H*} + \\ &\quad rg^H(p^m)(1-\pi)R^{H*} + (1-G^H(p^m))(1-\pi)R^{H*}\frac{dR^{H*}}{dp} / \frac{dR^{H*}}{dr} \\ &= \frac{\partial \left(p(1-G^H(p))\right)}{\partial p}|_{p=p^m} \times (1-\pi)\alpha^H R^{H*} + \\ &\quad R^{H*}\left[rg^H(p^m)(1-\pi) - (1-G^H(p^m))(1-\pi)h^H(p^m)r\right] \\ &= \frac{\partial \left(p(1-G^H(p))\right)}{\partial p}|_{p=p^m} \times (1-\pi)\alpha^H R^{H*} > 0.\end{aligned} \tag{48}$$

where the last equality holds since $h^H(p) = \frac{g^H(p)}{1-G^H(p)}$. The difference from Proposition 2 is that the term $\frac{\partial\left(p(1-G^H(p))\right)}{\partial p}|_{p=p^m}$ remains in this case. This discrepancy appears because of the difference in the average demand and type H consumers' demand. We have

$$\frac{\partial \left(p(1-G^H(p))\right)}{\partial p}|_{p=p^m} = 1 - G^H(p^m) - p^m g^H(p^m)$$

thus, the value of the above is positive if

$$1 - G^H(p^m) - p^m g^H(p^m) > 1 - G(p^m) - p^m g(p^m) = 0,$$

or

$$\frac{1-G^H(p^m)}{g^H(p^m)} > p^m = \frac{1-G(p^m)}{g(p^m)}.$$

Thus, assuming that the hazard rates satisfy $\frac{g^H(p)}{1-G^H(p)} < \frac{g(p)}{1-G(p)}$ for all $p \in (0,1)$, we can show that the profit-maximizing price under referrals is higher than p^m.$\blacksquare$

Thus, if only single type of consumers get referrals, then some of the results in the basic model must be modified.